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A PROGRAM FOR THE STABILITY ANALYSIS
OF PIPE POISEUILLE FLOW

Richard Howard Johnston

NAVAL POSTGRADUATE SCHOOL

Monterey, California



THESIS

A PROGRAM FOR THE STABILITY ANALYSIS
OF
PIPE POISEUILLE FLOW

by

Richard Howard Johnston III

March 1976

Thesis Advisor:

T. H. Gawain

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A minor error in Harrison's numerical method was corrected. Moreover, it proved necessary to reanalyze the conditions on the axis of symmetry of the pipe. These changes finally made it possible to obtain reasonable results.

Owing to time limitations, the number of solutions obtained using the corrected program was sufficient only to confirm its general validity. However, the results obtained are significant in that they disclose instabilities which are known to exist but which have not been accounted for in previous theoretical investigations.

A Program for the Stability Analysis
of

Pipe Poiseuille Flow

by

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ABSTRACT

Recent research by Harrison on the stability of parallel flows resulted in a successful solution of plane Poiseuille flow but produced unexplained anomalies for pipe flow. The purpose of the research in this paper was to find and correct errors in Harrison's initial analysis of the pipe flow problem.

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TABLE OF SYMBOLS

All quantities below are in dimensionless form.

D, D^2, \dots	The partial derivatives with respect to r in cylindrical coordinates.
e	Base of natural logarithms. Exponentiation is also denoted by $\exp()$.
$\bar{e}_x, \bar{e}_r, \bar{e}_\theta$	Unit vectors along the x, r , and θ axes in cylindrical coordinates.
F, G, H	Components of the velocity vector potential defined in equation (2-1).
i	$+\sqrt{-1}$, the imaginary unit. Also used as an index in the finite difference mesh in Section IV.
n	The number of interior points in the finite difference mesh of Section IV. Also used as the imaginary part of B , i.e. $B=in$.
Re	Reynolds number based on the mean velocity and pipe radius.
t	Time

T, t, P, Q	Shorthand notation for commonly occurring groups of symbols defined in equations (2-18), (2-19), (3-42), and (3-43).
\bar{v}	Velocity vector of the perturbation flow defined by equation (2-3).
\bar{W}	Complex vector potential of perturbation velocity defined by equations (2-1) and (2-2).
x, r, θ	Cylindrical coordinates.
A	$A_R + iA_I$ Complex wave number of the perturbation in the x direction.
B	iB_I Complex wave number of the perturbation in the θ direction
γ	$\gamma_R + \gamma_I$ Complex frequency of the perturbation.
\bar{P}	The vorticity transport equation (2-5) expressed in abbreviated notation as defined by equation (2-6).
$\Gamma_x, \Gamma_r, \Gamma_\theta$	The components of \bar{P} in cylindrical coordinates defined by equation (2-6).
$\bar{\omega}$	Vorticity vector of the perturbation flow defined in equation (2-4).

- ▽ Linear vector operator (nabla).
- x Vector cross-product operator.
- [] Brackets enclosing a matrix.
- { } Brackets enclosing a column vector.

I. INTRODUCTION

The research reported in this thesis was undertaken in an attempt to isolate sources of error causing obviously incorrect computer solutions for the three-dimensional linearized vorticity transport equation for pipe Poiseuille flow. Solutions obtained using the theory and program developed in Ref. 1 had indicated decreasing flow stability with decreasing Reynolds number, a result which is clearly inconsistent with theory and experimental data.

Initial analysis of that program confirmed that it was basically correct as presented in Ref. 1, but the nature of the solutions obtained indicated one or more errors in the formulation of the problem to be solved. Analysis of this problem formulation was conducted in three phases. The first phase consisted of ensuring accurate expression of the vorticity transport equation in terms of a complex velocity vector potential. The second and most significant phase of the analysis required extensive examination of the conditions required to satisfy the vorticity transport equation at the singular point on the axis of symmetry of the pipe. The third phase entailed establishing a finite difference approximation of this equation and its boundary conditions for adaptation of the problem to a standard form for computer solution.

II. THE VORTICITY TRANSPORT EQUATION

The governing equations for laminar flow of an incompressible fluid with constant viscosity are the continuity equation and the Navier-Stokes equation. Taking the curl ($\nabla \times$) of the latter equation and introducing a perturbation velocity (\bar{v}) and vorticity ($\bar{\omega}$) leads to the linearized perturbation vorticity transport equation as derived in Appendix A of Ref. 1.

Expressed in terms of the complex velocity vector potential, \bar{W} , which has the form

$$\bar{W}(x, r, \theta, t) = (\bar{e}_x F(r) + \bar{e}_r G(r) + \bar{e}_\theta H(r)) e^X \quad (2-1)$$

where

$$X = Ax + B\theta + \gamma t \quad (2-2)$$

and

$$\bar{v} = \nabla \times \bar{W} \quad (2-3)$$

$$\bar{\omega} = \nabla \times \bar{v}, \quad (2-4)$$

the vorticity transport equation becomes three simultaneous fourth-order differential equations of the form

$$\begin{aligned} & [M_4] \begin{Bmatrix} D^4 F \\ D^4 G \\ D^4 H \end{Bmatrix} + [M_3] \begin{Bmatrix} D^3 F \\ D^3 G \\ D^3 H \end{Bmatrix} + ([M_2] + \gamma [N_2]) \begin{Bmatrix} D^2 F \\ D^2 G \\ D^2 H \end{Bmatrix} + \\ & ([M_1] + \gamma [N_1]) \begin{Bmatrix} DF \\ DG \\ DH \end{Bmatrix} + ([M_0] + \gamma [N_0]) \begin{Bmatrix} F \\ G \\ H \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \end{aligned} \quad (2-5)$$

as derived in Appendix E of Ref. 1.

Equations (2-5) may be expressed in the abbreviated form

$$\bar{\Gamma} = \begin{Bmatrix} \Gamma_x \\ \Gamma_r \\ \Gamma_\theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (2-6)$$

where $\bar{\Gamma}$ appears to be a set of three coupled equations in the components of \bar{W} . As shown in Appendix B of Ref. 1, equations (2-6) actually express only two independent conditions. Therefore, an appropriate linear combination of Γ_x and Γ_θ allows expression of equations (2-5) as a set of two equations in three unknowns. The linear combination

$$\frac{-B}{r}\Gamma_x + A\Gamma_\theta = 0 \quad (2-7)$$

does not, in general, uncouple $\bar{\Gamma}$ but does reduce the highest order derivative of the component $G(r)$ in equations (2-5) to second order.

In a manner similar to that just described for $\bar{\Gamma}$, Appendix C of Ref. 1 illustrates the redundancy of the three components of \bar{W} . This allows arbitrary selection of one of these components as being uniformly zero for all r . As will be seen later, the maximum benefits from the consequences of the linear combination expressed by equation (2-7) are obtained if

$$F(r) = 0. \quad (2-8)$$

Incorporation of equations (2-7) and (2-8) in equation (2-5) results in expression of the vorticity transport equation in the form

$$\begin{aligned} & [M'_4] \begin{Bmatrix} D^4_G \\ D^4_H \end{Bmatrix} + [M'_3] \begin{Bmatrix} D^3_G \\ D^3_H \end{Bmatrix} + ([M'_2] - \gamma[N'_2]) \begin{Bmatrix} D^2_G \\ D^2_H \end{Bmatrix} + \\ & ([M'_1] - \gamma[N'_1]) \begin{Bmatrix} DG \\ DH \end{Bmatrix} + ([M'_0] - \gamma[N'_0]) \begin{Bmatrix} G \\ H \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \end{aligned} \quad (2-9)$$

where

$$[M'_4] = \begin{bmatrix} 0 & \frac{-A}{Re} \\ 0 & 0 \end{bmatrix} \quad (2-10)$$

$$[M'_3] = \begin{bmatrix} 0 & \frac{-2A}{rRe} \\ 0 & \frac{B}{rRe} \end{bmatrix} \quad (2-11)$$

$$[M'_2] = \begin{bmatrix} \frac{-4AB}{r^2 Re} & A \left(T + \frac{1}{Re} \left\{ \frac{3}{r^2} - t \right\} \right) \\ \frac{-t}{Re} & \frac{2B}{r^2 Re} \end{bmatrix} \quad (2-12)$$

$$[M'_1] = \begin{bmatrix} \frac{4AB}{r^3 Re} & \frac{A}{r} \left(\frac{1}{Re} \left\{ \frac{3}{r^2} [B^2 - 1] - A^2 \right\} + T \right) \\ \frac{1}{rRe} \left(\frac{B^2}{r^2} - A^2 \right) & \frac{-B}{r} \left(T + \frac{1}{r^2 Re} \right) \end{bmatrix} \quad (2-13)$$

$$[M'_0] = \begin{bmatrix} \frac{2AB}{r^2} \left(T - \frac{1}{Re} \left\{ t + \frac{2}{r^2} \right\} \right) & A \left(Tt + \frac{1}{r^2} \left\{ \frac{1}{Re} \left[t + \frac{3}{r^2} \right] - T \right\} \right) \\ Tt + \frac{1}{r^2 Re} \left(A^2 - \frac{B^2}{r^2} \right) & B \left(\frac{1}{r^2} \left\{ \frac{1}{Re} \left[2A^2 + \frac{1}{r^2} \right] - T \right\} - 4A \right) \end{bmatrix} \quad (2-14)$$

$$[N'_2] = \begin{bmatrix} 0 & -A \\ 0 & 0 \end{bmatrix} \quad (2-15)$$

$$[N'_1] = \begin{bmatrix} 0 & \frac{-A}{r} \\ 0 & \frac{B}{r} \end{bmatrix} \quad (2-16)$$

$$[N'_0] = \begin{bmatrix} \frac{-2AB}{r^2} & A \left(\frac{1}{r^2} - t \right) \\ -t & \frac{B}{r^2} \end{bmatrix} \quad (2-17)$$

and

$$t \cong A^2 + \frac{B^2}{r^2} \quad (2-18)$$

$$T \cong 2A(1 - r^2) - \frac{t}{Re} \quad (2-19)$$

This statement of the vorticity transport equation is identical to that published in Ref. 1. Note that equation (2-8) makes it unnecessary to include $F(r)$ and its derivatives in equations (2-9). The coefficients of this function and its derivatives are included in the arrays represented by equations (2-15) through (2-22) of Ref. 1 and have been verified. The reader's attention is called to a misprint occurring in Ref. 1 concerning the development to this point. Equation (2-22) of Ref. 1 should be corrected to read as follows.

$$[N_0] = \begin{bmatrix} \frac{-Bt}{r} & \frac{2AB}{r^2} & A \left(t - \frac{1}{r^2} \right) \\ 0 & t & \frac{-B}{r^2} \end{bmatrix} \quad (2-20)$$

III. BOUNDARY AND SPECIAL CONDITIONS

A. BOUNDARY CONDITIONS AT THE WALL

The boundary conditions at the wall, $r=1$, are derived from the fact that the axial, radial, and angular components of the flow velocity at the wall are identically zero. These components are obtained respectively from the three components of the velocity vector resulting from equation (2-3).

As shown in part two of Appendix F in Ref 1 for the case $F(r)=0$, these components imply the following boundary conditions at the wall.

$$G(1) = 0 \quad (3-1a)$$

$$H(1) = 0 \quad (3-1b)$$

$$DH(1) = 0 \quad (3-1c)$$

B. CONDITIONS ON THE AXIS OF SYMMETRY

It is important to note that the line of points constituting the axis of symmetry is not a boundary but rather a line consisting of an infinite number of singular points in the flow.

It should be noted in passing that all functions are periodic with respect to θ . Consequently, as shown in part one of Appendix G in Ref. 1, the wave number of the perturbation, B , must be a pure imaginary number such that

$$B = ni, \quad n = 0, 1, 2, \dots \quad (3-2)$$

Hence references to B in this paper shall be understood to indicate a pure imaginary number.

The primary condition to be imposed is that the vorticity transport equation be satisfied at the point $r=0$. Inspection of equations (2-9)

through (2-19) shows that, in general, these are not satisfied for direct substitution of $r=0$. To resolve this situation, consider the following development.

The coefficients of the functions $G(r)$ and $H(r)$ and their derivatives with respect to r up to fourth order in equations (2-9) through (2-19) may be expressed in the following form.

$$[M'_4] = [M_{4-0}] \quad (3-3)$$

$$[M'_3] = [M_{3-1}] 1/r \quad (3-4)$$

$$[M'_2] = [M_{2-2}] 1/r^2 + [M_{2-0}] + [M_{2+2}] r^2 \quad (3-5)$$

$$[M'_1] = [M_{1-3}] 1/r^3 + [M_{1-1}] 1/r + [M_{1+1}] r \quad (3-6)$$

$$[M'_0] = [M_{0-4}] 1/r^4 + [M_{0-2}] 1/r^2 + [M_{0-0}] + [M_{0+2}] r^2 \quad (3-7)$$

$$[N'_2] = [N_{2-0}] \quad (3-8)$$

$$[N'_1] = [N_{1-1}] 1/r \quad (3-9)$$

$$[N'_0] = [N_{0-2}] 1/r^2 + [N_{0-0}] \quad (3-10)$$

where

$$[M_{4-0}] = \begin{bmatrix} 0 & \frac{-A}{Re} \\ 0 & 0 \end{bmatrix} \quad (3-11)$$

$$[M_{3-1}] = \begin{bmatrix} 0 & \frac{-2A}{Re} \\ 0 & \frac{B}{Re} \end{bmatrix} \quad (3-12)$$

$$[M_{2-2}] = \begin{bmatrix} \frac{-4AB}{Re} & \frac{A}{Re} (3 - 2B^2) \\ \frac{-B^2}{Re} & \frac{2B}{Re} \end{bmatrix} \quad (3-13)$$

$$[M_{2-0}] = \begin{bmatrix} 0 & 2A^2 \left(1 - \frac{A}{Re}\right) \\ \frac{-A^2}{Re} & 0 \end{bmatrix} \quad (3-14)$$

$$[M_{2+2}] = \begin{bmatrix} 0 & -2A^2 \\ 0 & 0 \end{bmatrix} \quad (3-15)$$

$$[M_{1-3}] = \begin{bmatrix} \frac{4AB}{Re} & \frac{A}{Re} (2B^2 - 3) \\ \frac{B^2}{Re} & \frac{B}{Re} (B^2 - 1) \end{bmatrix} \quad (3-16)$$

$$[M_{1-1}] = \begin{bmatrix} 0 & 2A^2 \left(1 - \frac{A}{Re}\right) \\ \frac{-A^2}{Re} & -AB \left(2 - \frac{A}{Re}\right) \end{bmatrix} \quad (3-17)$$

$$[M_{1+1}] = \begin{bmatrix} 0 & -2A^2 \\ 0 & 2AB \end{bmatrix} \quad (3-18)$$

$$[M_{0-4}] = \begin{bmatrix} \frac{-4AB(B^2 + 1)}{Re} & \frac{A}{Re} (3 + B^2(2 - B^2)) \\ \frac{-B^2(B^2 + 1)}{Re} & \frac{B}{Re} (B^2 + 1) \end{bmatrix} \quad (3-19)$$

$$[M_{0-2}] = \begin{bmatrix} 4A^2B \left(1 - \frac{A}{Re}\right) & 2A^2 \left(1 - \frac{A}{Re}\right) (B^2 - 1) \\ 2AB^2 \left(1 - \frac{A}{Re}\right) + \frac{A^2}{Re} & AB \left(\frac{3A}{Re} - 2\right) \end{bmatrix} \quad (3-20)$$

$$[M_{0-0}] = \begin{bmatrix} -4A^2B & A^2 \left(2(1 - B^2) + A^2 \left\{2 - \frac{A}{Re}\right\}\right) \\ A \left(A^2 \left\{2 - \frac{A}{Re}\right\} - 2B^2\right) & -2AB \end{bmatrix} \quad (3-21)$$

$$[M_{0+2}] = \begin{bmatrix} 0 & -2A^4 \\ -2A^3 & 0 \end{bmatrix} \quad (3-22)$$

$$\begin{bmatrix} N_{2-0} \end{bmatrix} = \begin{bmatrix} 0 & -A \\ 0 & 0 \end{bmatrix} \quad (3-23)$$

$$\begin{bmatrix} N_{1-1} \end{bmatrix} = \begin{bmatrix} 0 & -A \\ 0 & B \end{bmatrix} \quad (3-24)$$

$$\begin{bmatrix} N_{0-2} \end{bmatrix} = \begin{bmatrix} -2AB & A(1 - B^2) \\ -B^2 & B \end{bmatrix} \quad (3-25)$$

$$\begin{bmatrix} N_{0-0} \end{bmatrix} = \begin{bmatrix} 0 & -A^3 \\ -A^2 & 0 \end{bmatrix} \quad (3-26)$$

The Maclaurin series representation of the functions $G(r)$ and $H(r)$ and their derivatives may be expressed according to the following scheme.

$$\begin{Bmatrix} G \\ H \end{Bmatrix} = \begin{bmatrix} G(r) + DG(r) + D^2G(r) + D^3G(r) + \dots \\ H(r) + DH(r) + D^2H(r) + D^3H(r) + \dots \end{bmatrix} \begin{Bmatrix} V \end{Bmatrix} \quad (3-27)$$

$$\begin{Bmatrix} DG \\ DH \end{Bmatrix} = \begin{bmatrix} DG(r) + D^2G(r) + D^3G(r) + D^4G(r) + \dots \\ DH(r) + D^2H(r) + D^3H(r) + D^4H(r) + \dots \end{bmatrix} \begin{Bmatrix} V \end{Bmatrix} \quad (3-28)$$

$$\begin{Bmatrix} D^2G \\ D^2H \end{Bmatrix} = \begin{bmatrix} D^2G(r) + D^3G(r) + D^4G(r) + D^5G(r) + \dots \\ D^2H(r) + D^3H(r) + D^4H(r) + D^5H(r) + \dots \end{bmatrix} \begin{Bmatrix} V \end{Bmatrix} \quad (3-29)$$

where

$$\begin{Bmatrix} V \end{Bmatrix} = \begin{Bmatrix} 1 \\ r \\ r^2/2! \\ r^3/3! \\ r^4/4! \\ \vdots \end{Bmatrix} \quad (3-30)$$

Substitution of equations (3-3) through (3-30) into equation (2-9) converts the vorticity transport equation to the following form.

$$\begin{aligned}
& \left[M_{4-0} \right] \begin{bmatrix} D^4 G(r) + \dots \\ D^4 H(r) + \dots \end{bmatrix} \begin{Bmatrix} 1 \\ \vdots \end{Bmatrix} + \left[M_{3-1} \right] \begin{bmatrix} D^3 G(r) + D^4 G(r) + \dots \\ D^3 H(r) + D^4 H(r) + \dots \end{bmatrix} \begin{Bmatrix} 1/r \\ 1 \\ \vdots \end{Bmatrix} \\
& + \left[M_{2-2} \right] \begin{bmatrix} D^2 G(r) + D^3 G(r) + D^4 G(r) + \dots \\ D^2 H(r) + D^3 H(r) + D^4 H(r) + \dots \end{bmatrix} \begin{Bmatrix} 1/r^2 \\ 1/r \\ 1/2 \\ \vdots \end{Bmatrix} \\
& + \left[M_{2-0} \right] \begin{bmatrix} D^2 G(r) + \dots \\ D^2 H(r) + \dots \end{bmatrix} \begin{Bmatrix} 1 \\ \vdots \end{Bmatrix} + \left[M_{2+2} \right] \begin{bmatrix} D^2 G(r) + \dots \\ D^2 H(r) + \dots \end{bmatrix} \begin{Bmatrix} r^2 \\ \vdots \end{Bmatrix} \\
& + \left[M_{1-3} \right] \begin{bmatrix} DG(r) + D^2 G(r) + D^3 G(r) + D^4 G(r) + \dots \\ DH(r) + D^2 H(r) + D^3 H(r) + D^4 H(r) + \dots \end{bmatrix} \begin{Bmatrix} 1/r^3 \\ 1/r^2 \\ 1/2r \\ 1/6 \\ \vdots \end{Bmatrix} \\
& + \left[M_{1-1} \right] \begin{bmatrix} DG(r) + D^2 G(r) + \dots \\ DH(r) + D^2 H(r) + \dots \end{bmatrix} \begin{Bmatrix} 1/r \\ 1 \\ \vdots \end{Bmatrix} + \left[M_{1+1} \right] \begin{bmatrix} DG(r) + \dots \\ DH(r) + \dots \end{bmatrix} \begin{Bmatrix} r \\ \vdots \end{Bmatrix}
\end{aligned}$$

$$\begin{aligned}
& + \left[M_{0-4} \right] \begin{bmatrix} G(r) + DG(r) + D^2G(r) + D^3G(r) + D^4G(r) + \dots \\ H(r) + DH(r) + D^2H(r) + D^3H(r) + D^4H(r) + \dots \end{bmatrix} \begin{Bmatrix} 1/r^4 \\ 1/r^3 \\ 1/2r^2 \\ 1/6r \\ 1/24 \\ \vdots \end{Bmatrix} \\
& + \left[M_{0-2} \right] \begin{bmatrix} G(r) + DG(r) + D^2G(r) + \dots \\ H(r) + DH(r) + D^2H(r) + \dots \end{bmatrix} \begin{Bmatrix} 1/r^2 \\ 1/r \\ 1/2 \\ \vdots \end{Bmatrix} \\
& + \left[M_{0-0} \right] \begin{bmatrix} G(r) + \dots \\ H(r) + \dots \end{bmatrix} \begin{Bmatrix} 1 \\ \vdots \end{Bmatrix} + \left[M_{0+2} \right] \begin{bmatrix} G(r) + \dots \\ H(r) + \dots \end{bmatrix} \begin{Bmatrix} r^2 \\ \vdots \end{Bmatrix} \\
& - \gamma \left[N_{2-0} \right] \begin{bmatrix} D^2G(r) + \dots \\ D^2H(r) + \dots \end{bmatrix} \begin{Bmatrix} 1 \\ \vdots \end{Bmatrix} - \gamma \left[N_{1-1} \right] \begin{bmatrix} DG(r) + \dots \\ DH(r) + \dots \end{bmatrix} \begin{Bmatrix} 1/r \\ 1 \\ \vdots \end{Bmatrix} \\
& - \gamma \left[N_{0-2} \right] \begin{bmatrix} G(r) + DG(r) + D^2G(r) + \dots \\ H(r) + DH(r) + D^2H(r) + \dots \end{bmatrix} \begin{Bmatrix} 1/r^2 \\ 1/r \\ 1/2 \\ \vdots \end{Bmatrix} \\
& - \left[N_{0-0} \right] \begin{bmatrix} G(r) + \dots \\ H(r) + \dots \end{bmatrix} \begin{Bmatrix} 1 \\ \vdots \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3-31}
\end{aligned}$$

Since equations (3-31) must be satisfied at all points in the flow, they must be satisfied in the limit as r approaches zero. A method of

determining the conditions to be satisfied at the singular point, $r=0$, is described below.

Noting that as r approaches zero those terms in equations (3-31) containing r to the first power or greater may be neglected, these equations may be regrouped as a power series in r and expressed in the following abbreviated form.

$$\begin{aligned}
 & [C_1] \begin{Bmatrix} G(0) \\ H(0) \end{Bmatrix} r^{-4} + [C_2] \begin{Bmatrix} DG(0) \\ DH(0) \end{Bmatrix} r^{-3} \\
 & + \left([C_3] \begin{Bmatrix} D^2G(0) \\ D^2H(0) \end{Bmatrix} + [C_4] \begin{Bmatrix} G(0) \\ H(0) \end{Bmatrix} \right) r^{-2} \\
 & + \left([C_5] \begin{Bmatrix} D^3G(0) \\ D^3H(0) \end{Bmatrix} + [C_6] \begin{Bmatrix} DG(0) \\ DH(0) \end{Bmatrix} \right) r^{-1} \\
 & + \left([C_7] \begin{Bmatrix} D^4G(0) \\ D^4H(0) \end{Bmatrix} + [C_8] \begin{Bmatrix} D^2G(0) \\ D^2H(0) \end{Bmatrix} + [C_9] \begin{Bmatrix} G(0) \\ H(0) \end{Bmatrix} \right) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (3-32)
 \end{aligned}$$

where

$$[C_1] = [M_{0-4}] \quad (3-33)$$

$$= \begin{bmatrix} \frac{-4ABP}{Re} & \frac{-AP(P-4)}{Re} \\ \frac{-B^2P}{Re} & \frac{BP}{Re} \end{bmatrix}$$

$$[C_2] = [M_{1-3}] + [M_{0-4}] \quad (3-34)$$

$$= \begin{bmatrix} \frac{-4AB^3}{Re} & \frac{-AB^2(P-5)}{Re} \\ \frac{-B^4}{Re} & \frac{2B^3}{Re} \end{bmatrix}$$

$$[C_3] = [M_{2-2}] + [M_{1-3}] + (1/2)[M_{0-4}] \quad (3-35)$$

$$= \begin{bmatrix} \frac{-2ABP}{Re} & \frac{-AP(P-4)}{2Re} \\ \frac{-B^2P}{2Re} & \frac{3BP}{2Re} \end{bmatrix}$$

$$[C_4] = [M_{0-2}] - \gamma [N_{0-2}] \quad (3-36)$$

$$= \begin{bmatrix} 2ABQ & AQ(P-2) \\ B^2Q + \frac{A^2}{Re} & -B\left(Q - \frac{A^2}{Re}\right) \end{bmatrix}$$

$$[C_5] = [M_{3-1}] + [M_{2-2}] + (1/2)[M_{1-3}] + (1/6)[M_{0-4}] \quad (3-37)$$

$$= \begin{bmatrix} \frac{-2AB(P+3)}{3Re} & \frac{-AB^2(B^4+4)}{6Re} \\ \frac{-B^2(P+3)}{6Re} & \frac{2B(P+3)}{3Re} \end{bmatrix}$$

$$[C_6] = [M_{1-1}] + [M_{0-2}] - \gamma ([N_{1-1}] + [N_{0-2}]) \quad (3-38)$$

$$= \begin{bmatrix} 2ABQ & AB^2Q \\ B^2Q & -2BQ \end{bmatrix}$$

$$[C_7] = [M_{4-0}] + [M_{3-1}] + (1/2)[M_{2-2}] + (1/6)[M_{1-3}] + (1/24)[M_{0-4}]$$

$$= \begin{bmatrix} \frac{AB(P+8)}{6Re} & \frac{-A(P+8)(P+4)}{24Re} \\ \frac{-B^2(P+8)}{24Re} & \frac{5B(P+8)}{24Re} \end{bmatrix} \quad (3-39)$$

$$[C_8] = [M_{2-0}] + [M_{1-1}] + (1/2)[M_{0-2}] -$$

$$\gamma ([N_{2-0}] + [N_{1-1}] + (1/2)[N_{0-2}]) \quad (3-40)$$

$$= \begin{bmatrix} ABQ & \frac{AQ(P+2)}{2} \\ \frac{1}{2}\left(B^2Q - \frac{3A^2}{Re}\right) & \frac{-B}{2}\left(3Q + \frac{A^2}{Re}\right) \end{bmatrix}$$

$$\begin{aligned}
[C_9] &= [M_{0-0}] - \gamma [N_{0-0}] \\
&= \begin{bmatrix} -4AB^2 & A^2 \left(A \left\{ Q + \frac{A^2}{Re} \right\} - 2(P - 2) \right) \\ A \left(A \left\{ Q + \frac{A^2}{Re} \right\} - 2B^2 \right) & -2AB \end{bmatrix}
\end{aligned} \tag{3-41}$$

and

$$P = B^2 + 1 \tag{3-42}$$

$$Q = 2A \left(1 - \frac{A}{Re} \right) + \gamma \tag{3-43}$$

As a consequence of equations (3-32), the following conditions must be met in the limit as r approaches zero.

$$[C_1] \begin{Bmatrix} G(0) \\ H(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3-44}$$

$$[C_2] \begin{Bmatrix} DG(0) \\ DH(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3-45}$$

$$[C_3] \begin{Bmatrix} D^2G(0) \\ D^2H(0) \end{Bmatrix} + [C_4] \begin{Bmatrix} G(0) \\ H(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3-46}$$

$$[C_5] \begin{Bmatrix} D^3G(0) \\ D^3H(0) \end{Bmatrix} + [C_6] \begin{Bmatrix} DG(0) \\ DH(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3-47}$$

$$[C_7] \begin{Bmatrix} D^4G(0) \\ D^4H(0) \end{Bmatrix} + [C_8] \begin{Bmatrix} D^2G(0) \\ D^2H(0) \end{Bmatrix} + [C_9] \begin{Bmatrix} G(0) \\ H(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3-48}$$

As discussed in Section II, the vorticity transport equation as expressed by equations (2-9) through (2-19) is, in general, a coupled set of differential equations. However, if $B=0$ these equations do uncouple. For this special case, the first of equations (2-9) becomes a homogeneous fourth-order differential equation in $H(r)$ and the second becomes a homogeneous second-order differential equation in $G(r)$. Thus, the

conditions expressed by equations (3-44) through (3-48) may now be examined for the special case $B=0$ and the general case $B>0$.

1. $B=0$

Inspection of equations (3-45) and (3-47) shows that they are identically satisfied for all values of the applicable odd-order derivatives of $G(0)$ and $H(0)$ for this case. Since the remaining equations uncouple for this case, the conditions on $G(0)$ and $H(0)$ may be studied independently.

a. $H(0)$

Sequential inspection of the first of equations (3-44), (3-46), and (3-48) verifies that there are three conditions to be enforced. These three conditions are as follows.

$$H(0) = 0 \quad (3-49)$$

$$D^2 H(0) = 0 \quad (3-50)$$

$$D^4 H(0) = 0 \quad (3-51)$$

b. $G(0)$

Although the second of equations (3-44) is identically satisfied for arbitrary $G(0)$, sequential inspection of the second of equations (3-46) and (3-48) yields the following two conditions which must be enforced.

$$G(0) = 0 \quad (3-52)$$

$$D^2 G(0) = 0 \quad (3-53)$$

2. $B>0$

With the exception of improbable special cases for which the determinants of any or all of the arrays $[C_1]$ through $[C_9]$ may be zero, the conditions to be met for this case are as follows.

$$G(0) = H(0) = 0 \quad (3-54)$$

$$DG(0) = DH(0) = 0 \quad (3-55)$$

$$D^2G(0) = D^2H(0) = 0 \quad (3-56)$$

$$D^3G(0) = D^3H(0) = 0 \quad (3-57)$$

$$D^4G(0) = D^4H(0) = 0 \quad (3-58)$$

The conditions expressed above represent a marked departure from the conditions previously thought to exist at $r=0$. Before the situation was properly understood, it was thought that the required conditions could be deduced from considerations of single-valuedness at this point. This approach is discussed in part two of Appendix G in Ref. 1 but must now be discarded as insufficient. In the interest of accuracy, the reader's attention is called to an error in equation (G-29) of that development. That equation should be corrected to read as follows.

$$v_1(0) = -iw_1(0) \quad (3-59)$$

IV. NUMERICAL METHODS

Regrouping equations (2-9) allows expression of the equations to be solved in the following format.

$$\begin{aligned}
 & \left[M'_4 \right] \begin{Bmatrix} D^4 G \\ D^4 H \end{Bmatrix} + \left[M'_3 \right] \begin{Bmatrix} D^3 G \\ D^3 H \end{Bmatrix} + \left[M'_2 \right] \begin{Bmatrix} D^2 G \\ D^2 H \end{Bmatrix} + \left[M'_1 \right] \begin{Bmatrix} DG \\ DH \end{Bmatrix} + \left[M'_0 \right] \begin{Bmatrix} G \\ H \end{Bmatrix} \\
 & - \delta \left(\left[N'_2 \right] \begin{Bmatrix} D^2 G \\ D^2 H \end{Bmatrix} + \left[N'_1 \right] \begin{Bmatrix} DG \\ DH \end{Bmatrix} + \left[N'_0 \right] \begin{Bmatrix} G \\ H \end{Bmatrix} \right) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (4-1)
 \end{aligned}$$

The coefficient arrays are defined by equations (2-10) through (2-19).

As discussed in Section III, the nature of this system of equations depends on the value of the complex perturbation wave number, B . For the special case $B=0$, the first of equations (4-1) becomes a homogeneous fourth-order differential equation in $H(r)$ and the second becomes a homogeneous second-order differential equation in $G(r)$. For the general case of $B>0$, equations (4-1) do not uncouple and must be solved simultaneously.

An additional influence of B is reflected in the character of the conditions at the singular point $r=0$. These conditions, as developed in Section III, are summarized below.

For the special case $B=0$, the first of equations (4-1) must simultaneously satisfy the conditions

$$H(0) = 0 \quad (4-2)$$

$$D^2 H(0) = 0 \quad (4-3)$$

$$D^4 H(0) = 0 \quad (4-4)$$

and the second of equations (4-1) must similarly satisfy the following conditions.

$$G(0) = 0 \quad (4-5)$$

$$D^2G(0) = 0 \quad (4-6)$$

For the general case where $B > 0$, equations (4-1) must simultaneously satisfy the conditions below.

$$G(0) = H(0) = 0 \quad (4-7)$$

$$DG(0) = DH(0) = 0 \quad (4-8)$$

$$D^2G(0) = D^2H(0) = 0 \quad (4-9)$$

$$D^3G(0) = D^3H(0) = 0 \quad (4-10)$$

$$D^4G(0) = D^4H(0) = 0 \quad (4-11)$$

The boundary conditions at the wall, $r=1$, are a consequence only of zero-velocity viscous effects at that point and thus do not vary with B . These conditions are as follows.

$$G(1) = 0 \quad (4-12)$$

$$H(1) = 0 \quad (4-13)$$

$$DH(1) = 0 \quad (4-14)$$

Using the method of finite differences, the functions $G(r)$ and $H(r)$ may be approximated by a finite number of discrete, evenly spaced unknowns. As shown in Figure 4-1 below, the non-dimensionalized radius of the pipe may be divided into a one-dimensional computational mesh of uniform spacings consisting of n interior points, $n+1$ intervals, and $n+2$ total points, including the boundary point at $r=1$ and the singular point at $r=0$. The uniform spacing between each of the mesh points is δ as defined below.

$$\delta = 1/(n+1) \quad (4-15)$$

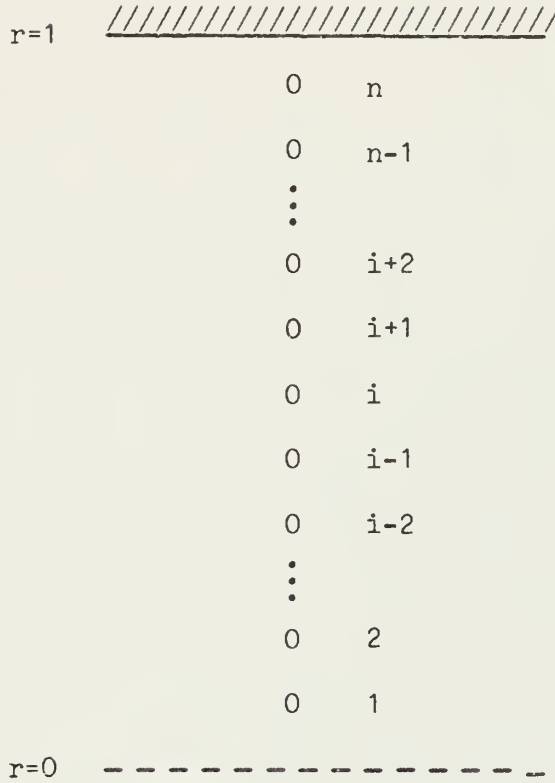


Figure 4-1 Finite Difference Mesh

By taking the Taylor series expansion of the function $H(r)$ about the i^{th} mesh point in terms of the values of this function at the mesh points $i+2$, $i+1$, $i-1$, and $i-2$ the second-order central difference approximations of the derivatives of $H(r)$ at the i^{th} point are found to have the following form.

$$DH_i = (H_{i+1} - H_{i-1})/2\delta \quad (4-16)$$

$$D^2H_i = (H_{i+1} - 2H_i + H_{i-1})/\delta^2 \quad (4-17)$$

$$D^3H_i = (H_{i+2} - 2(H_{i+1} - H_{i-1}) - H_{i-2})/2\delta^3 \quad (4-18)$$

$$D^4H_i = (H_{i+2} - 4H_{i+1} + 6H_i - 4H_{i-1} + H_{i-2})/\delta^4 \quad (4-19)$$

where

$$i = n, n-1, \dots, 3, 2, 1 \quad (4-20)$$

The approximations for the derivatives of $G(r)$ are determined in an identical manner. The error of equations (4-16) through (4-19) is of the order of magnitude δ^2 . The order of magnitude of this error may be reduced by expanding the method of derivation of these equations to include more adjacent points on either side of the i^{th} , central, point.

It is important to note a discrepancy appearing in Ref. 1 concerning the development to this point. Careful comparison of Figure 3-1 in Ref. 1 with Figure 4-1 in this paper reveals that the direction of the labeling schemes used to depict the finite difference mesh has been reversed. Note also that equations (3-2) in Ref. 1 compare exactly with equations (4-16) through (4-19) above. When these equations are used in conjunction with the labeling scheme used in Figure 3-1 in Ref. 1, the signs of the odd-order derivative approximations are reversed. This was a major factor in producing the erroneous solutions obtained from the program presented in that paper.

Substitution of the central difference approximations for the derivatives of the functions G and H in each of equations (4-1) results in a set of n linear, algebraic difference equations in terms of the unknown values for each of these functions at each of the n interior points of the finite difference mesh depicted in Figure 4-1. Since each of these equations is of the form of a linear combination of the i^{th} , central, mesh point and the two adjacent points on either side of this central point, this system of equations consists of banded coefficient arrays multiplying vectors containing the unknown values of the functions at each of the n interior points. By using this technique the

problem is converted to an eigenvalue problem of the general form

$$[X]\{V\} - \lambda[Y]\{V\} = \{0\} \tag{4-21}$$

with the basic composition of the arrays $[X]$ and $[Y]$ and the vector $\{V\}$ as illustrated in Figure 4-2 below.

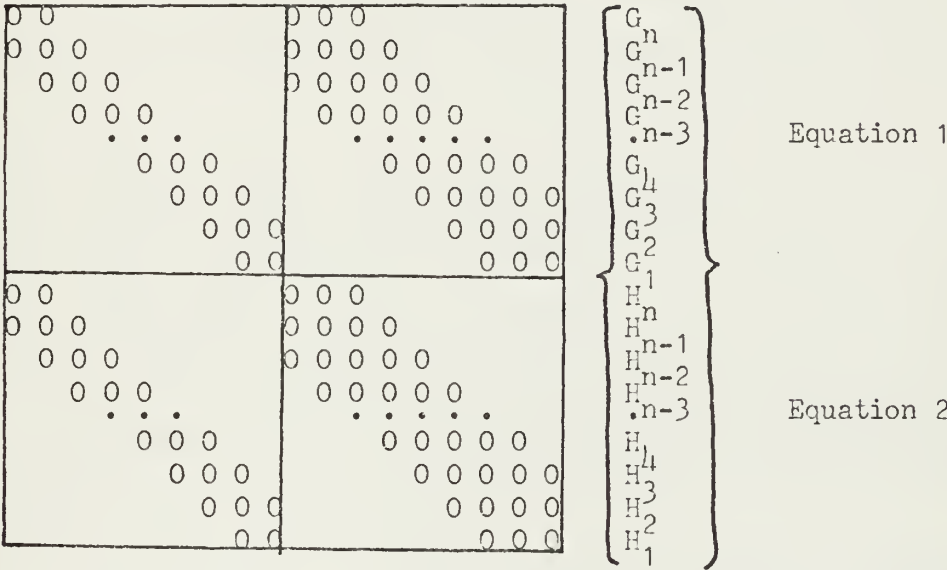


Figure 4-2 Basic Composition of the Coefficient Arrays and the Vector of Unknowns.

The exact composition of $[X]$ and $[Y]$ depends upon the value of B . These arrays are established by the subroutine MSET2 in conjunction with the function subprograms CHM1E1 and CHM1E2, which compute the numerical value for each element in these arrays. The function subprogram CHM1E1 provides those coefficients required from the first of equations (4-1) and CHM1E2 provides those coefficients required from the second of these equations. For the special case $B=0$, all coefficients contained in the upper right and lower left quadrants of Figure 4-2 would be zero.

Of particular interest are the difference equations whose central points are adjacent to the boundary at $r=1$, point n , and the singular point at $r=0$, point 1. To amplify this matter, consider the homogeneous differential equation in $H(r)$ resulting from setting $B=0$ in equations (4-1). Use of equation (4-19) to evaluate the fourth derivative of this function at point n of the finite difference mesh reveals that

$$D^4 H_n = (H_{n+2} - 4H_{n+1} + 6H_n - 4H_{n-1} + H_{n-2})/\delta^4. \quad (4-22)$$

Similarly, for $i=1$ equation (4-19) becomes

$$D^4 H_1 = (H_3 - 4H_2 + 6H_1 - 4H_0 + H_{-1})/\delta^4. \quad (4-23)$$

Since $0 \leq r \leq 1$, equations (4-22) and (4-23) present apparent problems in that H_{n+2} is located at $r=1+\delta$ and H_{-1} is located at $r=-\delta$, i.e. beyond the allowable range of values for r .

These inconsistencies are resolved using the boundary conditions expressed by equations (4-12) through (4-14) and the conditions at $r=0$ expressed by equations (4-2) through (4-4). Using the labeling convention of Figure 4-1 and equation (4-13) it is easily confirmed that

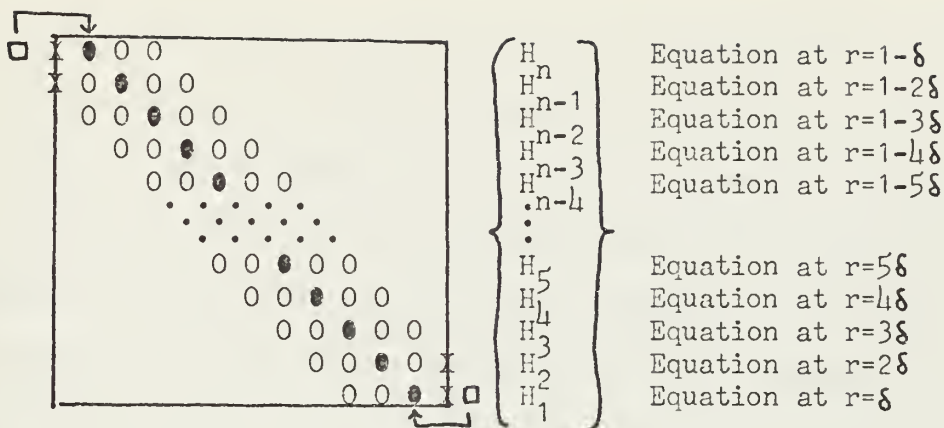
$$H(1) = H_{n+1} = 0. \quad (4-24)$$

Comparison of equations (4-14) and (4-16) yields the following relationships.

$$DH(1) = DH_{n+1} = (H_{n+2} - H_n)/2 = 0 \quad (4-25)$$

$$H_{n+2} = H_n. \quad (4-26)$$

Equation (4-26) expresses the virtual point at $r=1+\delta$ in terms of the interior point n . Equation (4-22) may now be expressed exclusively in terms of boundary and interior points as follows (see depiction of equation at $r=1-\delta$ in Figure 4-3 below).



- Central point of the difference equation.
- O Other points in the difference equation.
- X Boundary or singular point where the value of the variable is zero.
- Virtual point whose coefficient is combined with the coefficient of the point indicated by an arrow.

Figure 4-3 Illustration of Basic Method Used to Include Virtual Points in Difference Equations.

The virtual point at $r=-\delta$ can be expressed in terms of the interior point at $r=\delta$ in a similar manner. Expressed in terms of the labeling convention of Figure 4-1, equation (4-2) becomes

$$H_0 = 0. \quad (4-27)$$

Comparison of equations (4-3) and (4-17) shows that

$$D^2 H_0 = (H_1 - 2H_0 + H_{-1})/\delta^2 = 0. \quad (4-28)$$

Direct substitution of equation (4-27) in equation (4-28) implies

$$H_{-1} = -H_1. \quad (4-29)$$

This leads to a more appropriate expression of equation (4-23) in terms of interior points as follows.

$$D^4 H_1 = (H_3 - 4H_2 + \underbrace{6H_1 - 4H_0 - H_{-1}}_{\text{combined}})/\delta^4 \quad (4-30)$$

Consider now the homogeneous second-order differential equation in $G(r)$ for the special case $B=0$. Since the highest order derivative of

this function contained in equations (4-1) is second-order, inspection of equations (4-16) and (4-17) verifies the fact that no virtual points are required in the vicinities of the wall or the axis of symmetry.

As previously mentioned, the general case of $B > 0$ requires solution of equations (4-1) as a system of coupled equations. Implementation of the conditions at $r=0$ for this problem as given by equations (4-7) through (4-11) requires special attention.

The central difference approximation of the fourth derivative of H at $r=\delta$ requires a virtual point at $r=-\delta$. Expression of this virtual point in terms of the interior point at $r=\delta$ is complicated by the requirement that the five conditions for the function H given by equations (4-7) through (4-11) be satisfied simultaneously. Using equation (4-16) to satisfy equation (4-8) seems to imply that

$$H_{-1} = H_1 . \quad (4-31)$$

On the other hand, as previously shown in the development of equation (4-29)

$$H_{-1} = -H_1 . \quad (4-32)$$

This contradiction between equations (4-31) and (4-32) poses a problem. However, the Maclaurin series approximation for the function $H(r)$ and its derivatives in the vicinity of $r=0$ is helpful in resolving this contradiction. These approximations are as shown below.

$$H(r) = A_0 + A_1 r + A_2 \frac{r^2}{2!} + A_3 \frac{r^3}{3!} + A_4 \frac{r^4}{4!} + A_5 \frac{r^5}{5!} + O(r^6) \quad (4-33)$$

$$DH(r) = A_1 + A_2 r + A_3 \frac{r^2}{2!} + A_4 \frac{r^3}{3!} + A_5 \frac{r^4}{4!} + O(r^5) \quad (4-34)$$

$$D^2H(r) = A_2 + A_3 r + A_4 \frac{r^2}{2!} + A_5 \frac{r^3}{3!} + O(r^4) \quad (4-35)$$

$$D^3H(r) = A_3 + A_4 r + A_5 \frac{r^2}{2!} + O(r^3) \quad (4-36)$$

$$D^4H(r) = A_4 + A_5 r + O(r^2) \quad (4-37)$$

A comparison of equations (4-33) through (4-37) with equations (4-7) through (4-11) indicates that at $r=0$

$$A_0 = A_1 = A_2 = A_3 = A_4 = 0. \quad (4-38)$$

Substitution of equations (4-38) into equation (4-33) yields the following sixth-order approximation for $H(r)$ in the limit as r approaches zero.

$$H(r) = A_5 \frac{r^5}{5!} + O(r^6) \quad (4-39)$$

Evaluation of equation (4-39) for $r=\pm\delta$ yields

$$H(\delta) = A_5 \frac{\delta^5}{5!} \quad (4-40)$$

and

$$H(-\delta) = A_5 \frac{(-\delta)^5}{5!} \quad (4-41)$$

Inspection of equations (4-40) and (4-41) quickly confirms that enforcement of equations (4-7) through (4-11) requires that

$$H_{-1} = -H_1. \quad (4-42)$$

Note that the use of equation (4-39) as an approximation for the function $H(r)$ in equation (4-11) does not alter the second-order accuracy of the latter approximation. This may be conceptualized by a symbolic substitution of equation (4-39) in equation (4-11) as shown below.

$$D^4H_1 = ((H_3 - 4H_2 + 6H_1 - 4H_0 + H_{-1}) + O(\delta^6))/\delta^4 \quad (4-43)$$

$$= (H_3 - 4H_2 + 6H_1 - 4H_0 + H_{-1})/\delta^4 + O(\delta^2) \quad (4-44)$$

With the formulation of the problem as discussed to this point accomplished, i.e. in the form of equation (4-21), the method used for the remainder of the solution is summarized in the following steps:

- 1.) Subroutine CDMTIN inverts the second coefficient array in equation (4-21), array $[Y]$. CDMTIN was obtained from the IBM Library routine CMTRIN by modifying it to make it applicable to double precision arrays.
- 2.) Both coefficient arrays, $[X]$ and $[Y]$, are then premultiplied by $[Y]^{-1}$. Since multiplication of an array by its inverse invariably results in the identity matrix, $[I]$, only the product $[Y]^{-1}[X]$ is computed using subroutine MULM. This converts the eigenvalue problem of equation (4-21) to the more conventional form

$$([Z] - \lambda[I])\{V\} = \{0\} . \quad (4-45)$$

where

$$[Z] = [Y]^{-1}[X] \quad (4-46)$$

- 3.) Since all the programs available for solving equations (4-45) require that the real and imaginary parts of the elements of $[Z]$ be presented in separate arrays, subroutine DSPLIT is called to accomplish this.

- 4.) The eigenvalues of equations (4-45) are computed by subroutines EHESSC and ELRH1C which are available through the International Mathematical and Statistical Library. These subroutines reduce the matrix $[Z]$ into the complex upper Hessenberg form and then solve for the eigenvalues. The results are then passed back to the main program, R2.

- 5.) The eigenvalues thus returned are listed on the computer output and plotted on the complex plane using subroutine PLOTP from the IBM Library.

V. RESULTS

Owing to time constraints, the number of computer solutions obtained was sufficient only to confirm the general validity of the program presented in this paper. For this reason, a detailed discussion of flow stability will not be undertaken here. However, the following brief discussion should provide sufficient information for interpreting the data presented in this section.

The characteristics of the flow which govern the stability are defined by the input variables A , B , and Reynolds number. A complete set of eigenvalues, γ , is obtained for each chosen combination of these variables. Recalling the form of the perturbation velocity vector potential, \bar{W} , as presented in Section II,

$$\bar{W}(x,r,\theta,t) = (\bar{e}_x F(r) + \bar{e}_r G(r) + \bar{e}_\theta H(r)) \exp(Ax + B\theta + \gamma t) \quad (5-1)$$

where

$$\gamma = \gamma_R + \gamma_I, \quad (5-2)$$

it is readily seen that positive values for the real part, γ_R , of the complex perturbation frequency represent an exponential growth rate in time and negative values of γ_R represent an exponential decay rate. The algebraically greatest value of γ_R contained in the solution set is therefore used as the stability criterion. The corresponding eigenvalue is said to be the least stable root. The flow is considered to be stable, neutrally stable, or unstable with respect to stationary coordinates according to whether the least stable root is less than, equal to, or greater than zero, respectively. For a discussion of stability with respect to moving coordinates see part D of Section II in Ref. 1.

All computations were made using a half channel mesh containing 30 interior points and with A_I held constant at a value of one. Tables I through V present the real part of the least stable root for each of the solutions obtained using the program developed in this paper. In general, these solutions are encouraging in that those errors which had previously caused solutions to exhibit decreasing stability with decreasing Reynolds number seem to have been resolved. Results appear to be reasonable in that they exhibit stability characteristics which are qualitatively similar to those obtained for the plane flow problem as detailed in Section IV of Ref. 1. Specifically, for the range of Reynolds numbers investigated, all flows were stable for $A_R=0$ while instabilities were found for $A_R=-0.05$. These results are significant in that previous investigations of fully developed pipe flow have never accounted for the instabilities that are known to exist.

Re	$A_R = 0$	$A_R = -0.05$
100	$-3.44 \cdot 10^{-1}$	$-2.62 \cdot 10^{-1}$
200	$-2.88 \cdot 10^{-1}$	$-1.95 \cdot 10^{-1}$
500	$-1.80 \cdot 10^{-1}$	$-8.45 \cdot 10^{-2}$
1000	$-1.27 \cdot 10^{-1}$	$-2.99 \cdot 10^{-2}$
2000	$-8.93 \cdot 10^{-2}$	$8.57 \cdot 10^{-3}$
5000	$-5.61 \cdot 10^{-2}$	$4.26 \cdot 10^{-2}$
10000	$-3.95 \cdot 10^{-2}$	$5.96 \cdot 10^{-2}$

Table I γ_R for $B_I = 0$.

Re	$A_R = 0$	$A_R = -0.05$
100	$-3.05 \cdot 10^{-1}$	$-2.31 \cdot 10^{-1}$
200	$-2.73 \cdot 10^{-1}$	$-2.02 \cdot 10^{-1}$
500	$-1.84 \cdot 10^{-1}$	$-9.31 \cdot 10^{-2}$
1000	$-1.33 \cdot 10^{-1}$	$-3.98 \cdot 10^{-2}$
2000	$-9.58 \cdot 10^{-2}$	$-3.94 \cdot 10^{-4}$
5000	$-5.98 \cdot 10^{-2}$	$3.74 \cdot 10^{-2}$
10000	$-4.13 \cdot 10^{-2}$	$5.68 \cdot 10^{-2}$

Table II γ_R for $B_I = 1$

Re	$A_R = 0$	$A_R = -0.05$
100	$-4.91 \cdot 10^{-1}$	$-3.38 \cdot 10^{-1}$
200	$-3.36 \cdot 10^{-1}$	$-2.49 \cdot 10^{-1}$
500	$-2.09 \cdot 10^{-1}$	$-1.19 \cdot 10^{-1}$
1000	$-1.48 \cdot 10^{-1}$	$-5.48 \cdot 10^{-2}$
2000	$-1.05 \cdot 10^{-1}$	$-9.54 \cdot 10^{-3}$
5000	$-6.64 \cdot 10^{-2}$	$3.06 \cdot 10^{-2}$
10000	$-4.71 \cdot 10^{-2}$	$5.08 \cdot 10^{-2}$

Table III γ_R for $B_I = 2$

Re	$A_R = 0$	$A_R = -0.05$
100	$-4.94 \cdot 10^{-1}$	$-4.44 \cdot 10^{-1}$
200	$-4.27 \cdot 10^{-1}$	$-3.82 \cdot 10^{-1}$
500	$-2.89 \cdot 10^{-1}$	$-2.01 \cdot 10^{-1}$
1000	$-2.05 \cdot 10^{-1}$	$-1.13 \cdot 10^{-1}$
2000	$-1.45 \cdot 10^{-1}$	$-5.05 \cdot 10^{-2}$
5000	$-9.16 \cdot 10^{-2}$	$4.74 \cdot 10^{-3}$
10000	$-6.49 \cdot 10^{-2}$	$3.25 \cdot 10^{-2}$

Table IV γ_R for $B_I = 3$

Re	$A_R = 0$	$A_R = -0.05$
100	$-6.16 \cdot 10^{-1}$	$-5.70 \cdot 10^{-1}$
200	$-4.69 \cdot 10^{-1}$	$-4.39 \cdot 10^{-1}$
500	$-3.44 \cdot 10^{-1}$	$-2.87 \cdot 10^{-1}$
1000	$-2.75 \cdot 10^{-1}$	$-1.74 \cdot 10^{-1}$
2000	$-1.87 \cdot 10^{-1}$	$-9.36 \cdot 10^{-2}$
5000	$-1.18 \cdot 10^{-1}$	$-2.25 \cdot 10^{-2}$
10000	$-8.36 \cdot 10^{-2}$	$1.33 \cdot 10^{-2}$

Table V γ_R for $B_I = 4$


```

PROGRAM #R2
.....
PROGRAM TO PRINT EIGENVALUES
FOR THE 3-D CYLINDRICAL FLOW PROBLEM
.....
INPUT VALUES ARE N,REY,AR,AI,BR, AND BI.
(USING NAMELIST 'LIST')
.....
IMPLICIT REAL*8(A-H,O-Z)
CCOMPLEX*16 A,B,DERM,G
CCOMPLEX*16 XMAT(60,60),YMAT(60,60),WV(60)
CCOMPLEX*16 CFM1E1,CFM2E1,CFM1E2,CFM2E2,CGM1E1,CGM2E1,
* CGM1E2,CGM2E2,CHM1E1,CHM2E1,CHM1E2,CHM2E2
REAL*8 GR(60),GI(60)
REAL*4 GR4(60),GI4(60)
DIMENSION IVEC(100)
COMMON / COEFNT / A,B,G,REY,DELR
NAMELIST / LIST / N,REY,AR,AI,BR,BI
EXTERNAL CFM1E1,CFM2E1,CFM1E2,CFM2E2,CGM1E1,CGM2E1,
* CGM1E2,CGM2E2,CHM1E1,CHM2E1,CHM1E2,CHM2E2
.....
READ AND VERIFY THE INPUT VARIABLES.
.....
1 READ(5,LIST,END=100)
A = DCMPLX(AR,AI)
B = DCMPLX(BR,BI)
MCIN = 60
N = 30
WRITE(6,9000)
FCRMT(1,1)
9000 WRITE(6,9001)N,REY,A,B
9001 FCRMT(//,1X,'N=',14,/,1X,'REY=',F10.2,/,1X,'ALPHA=',2F12.7,8X,
* BETA=',2F12.7,/,1X,'FOR THE CASE F(R)=0.',/)
.....
SET UP THE CENTRAL DIFFERENCE APPROXIMATION AT EACH POINT IN
THE MESH FOR THOSE TERMS IN THE VORTICITY TRANSPORT EQUATION
WHICH CONTAIN GAMMA AS A FACTOR.
.....

```



```

CALL MSET2(YMAT,N,MDIM,CGM2E1,O,O)
CALL MSET2(YMAT,N,MDIM,CHM2E1,O,N)
CALL MSET2(YMAT,N,MDIM,CGM2E2,N,O)
CALL MSET2(YMAT,N,MDIM,CHM2E2,N,N)

```

INVERT THE RESULTING ARRAY.

```

CALL COMTIN(MDIM,YMAT,MDIM,DETERM)

```

SET UP THE CENTRAL DIFFERENCE APPROXIMATION AT EACH POINT IN THE MESH FOR THOSE TERMS IN THE VORTICITY TRANSPORT EQUATION WHICH DO NOT CONTAIN GAMMA AS A FACTOR.

```

CALL MSET2(XMAT,N,MDIM,CGM1E1,O,O)
CALL MSET2(XMAT,N,MDIM,CHM1E1,O,N)
CALL MSET2(XMAT,N,MDIM,CGM1E2,N,O)
CALL MSET2(XMAT,N,MDIM,CHM1E2,N,N)

```

PREMULTIPLY XMAT BY THE INVERSE OF YMAT TO CONVERT THE PROBLEM TO THE STANDARD FORM:

$$A \cdot X - \text{GAMMA} \cdot I = 0$$

```

CALL MULM(YMAT,XMAT,MDIM,MDIM,WV)

```

SPLIT THE RESULTING ARRAY INTO REAL (XMAT) AND IMAGINARY (YMAT) PARTS.

```

CALL DSPLIT(MDIM,MDIM,YMAT,XMAT,YMAT)

```

CALCULATE THE EIGENVALUES (GAMMA) FOR THE EQUATION.

```

CALL EHESSC(XMAT,YMAT,1,MDIM,MDIM,MDIM,IVEC)
CALL ELPHIC(XMAT,YMAT,1,MDIM,MDIM,MDIM,GR,CI,INERR,IER)
IF(INERR.NE.0) WRITE(6,9010) INERR,IER
FCRMAT('O* * * * *',J7,' ON EIGENVALUE',

```

9010 *

CC


```

C
C
C      WRITE THE RESULTS.
C
C      9002      WRITE(6,9002)
C                FCRMAT(//,16X,'GAMMA REAL',10X,'GAMMA IMAG')
C                DC 10 I = 1,MDIM
C                GR4(I) = SNGL(GR(I))
C                GI4(I) = SNGL(GI(I))
C      1003      WRITE(6,9003) I,GR(I),GI(I)
C                FORMAT('0',I10,1P2D20.10)
C
C      C
C      C      PLOT THE RESULTS.
C      C
C      C      9000      WRITE(6,9000)
C      C                CALL FLOIP(GR4,GI4,MDIM,0)
C      C                WRITE(6,9001) N,REY,A,B
C      C      100      GC TO 1
C      C                WRITE(6,9000)
C      C                STOP
C      C                END

```



```

C.....SUBROUTINE MSET2(X,N,MDIM,CFMAT,MV,MH).....
PURPOSE
MSET2 GENERATES THE ARRAYS REPRESENTING THE CENTRAL DIFFERENCE
APPROXIMATION OF THE VORTICITY TRANSPORT EQUATION IN TERMS OF
THE VELOCITY VECTOR POTENTIAL.

USAGE
CALL MSET2(X,N,MDIM,CFMAT,MV,MH)

DESCRIPTION OF PARAMETERS
X - THE NAME OF THE ARRAY BEING GENERATED. MUST BE DIMENSIONED
    IN THE CALLING PROGRAM
N- THE ROW DIMENSION OF THE MATRIX X. MUST BE .GE. N.
MDIM - THE COLUMN DIMENSION OF THE MATRIX X. MUST BE .GE. N.
CFMAT - THE NAME OF A FUNCTION SUBPROGRAM WITH TWO PARAMETERS, K
        AND R. INDICATING WHICH TERM OF THE CENTRAL DIFFERENCING
        SCHEME IS DESIRED, AND THE POSITION OF THE CENTRAL POINT
        RELATIVE TO THE AXIS OF THE PIPE. CFMAT MUST BE DECLARED
        EXTERNAL IN THE CALLING PROGRAM.

MV - ROW INDEX FOR INITIALIZING PLACEMENT OF ELEMENTS INTO
    MATRIX X.
MH - COLUMN INDEX FOR INITIALIZING PLACEMENT OF ELEMENTS INTO
    MATRIX X.

THE FOLLOWING IS OUTPUT BY MSET2
X - THE N BY N MATRIX INTO WHICH THE COEFFICIENTS OF THE CENTRAL
    DIFFERENCING ARE PUT.

OTHER ROUTINES NEEDED
FUNCTION SUBPROGRAM NAME PASSED IN THE CALLING PARAMETER 'CFMAT'
.....
SUBROUTINE MSET2(X,N,MDIM,CFMAT,MV,MH)
REAL*8 REY,R,DEL,DFLOAT
COMPLEX*16 A,B,G
COMMON / COEFNT / A,B,G,REY,DEL
COMPLEX*16 CFMAT

```



```

CCCCC CCMPLX*16 X(MDIM,MDIM)

        DEFINE THE SPACING OF THE INTERIOR MESH POINTS.

        DEL = 1D0/DFLOAT(N+1)

        CHECK IF MATRIX DIMENSIONED LARGE ENOUGH.

        IF(N+MV.LE.MDIM.AND.N+MH.LE.MDIM) GO TO 1
        WRITE(6,9000)
        9000 * FFORMAT(10* * * ERROR - ARRAYS NOT DIMENSIONED LARGE',
        * , ENOUGH * * *)
        STCP

        INITIALIZE ALL ELEMENTS IN THE ARRAY TO ZERO.

        1 DO 10 I=1,N
          DO 10 J=1,N
            10 X(I+MV,J+MH) = (0D0,0D0)

        ESTABLISH THE CENTRAL DIFFERENCE APPROXIMATION AT EACH POINT IN
        THE MESH.

        R = 1D0-DEL
        X(1+MV,1+MH) = CFMAT(3,R)+CFMAT(1,R)
        X(1+MV,2+MH) = CFMAT(4,R)
        X(1+MV,3+MH) = CFMAT(5,R)

        R = 1D0-2D0*DEL
        X(2+MV,1+MH) = CFMAT(2,R)
        X(2+MV,2+MH) = CFMAT(3,R)
        X(2+MV,3+MH) = CFMAT(4,R)
        X(2+MV,4+MH) = CFMAT(5,R)

        IL = N-2
        DO 20 I=3,IL
          K = I-3
          R = 1D0-DEL*DFLOAT(I)
          DO 20 J=1,5
            20 X(I+MV,K+J+MH) = CFMAT(J,R)

```


C

```
R = 2*DEL  
X(N-1+MV,N-3+MH) = CFMAT(1,R)  
X(N-1+MV,N-2+MH) = CFMAT(2,R)  
X(N-1+MV,N-1+MH) = CFMAT(3,R)  
X(N-1+MV,N+MH) = CFMAT(4,R)
```

C

```
R = DEL  
X(A+MV,N-2+MH) = CFMAT(1,R)  
X(A+MV,N-1+MH) = CFMAT(2,R)  
X(A+MV,N+MH) = CFMAT(3,R)+CFMAT(5,R)  
30 RETURN  
END
```


.....FUNCTION CHM1E1(K,R).....
 (POLAR COORDINATES)

PURPOSE

SECTION I

RETURNS THE VALUES FOR EACH OF THE COEFFICIENTS IN THE LINEAR COMBINATION OF THE FIRST AND THIRD EQUATIONS RESULTING FROM EXPRESSION OF THE VORTICITY TRANSPORT EQUATION IN TERMS OF THE VELOCITY VECTOR POTENTIAL.

DESCRIPTION OF PARAMETERS

EXAMPLE OF THE CALLING ARGUMENT:

C(F,G,H) (4,3,2,1,0) M(1,2)

C(F,G,H) - COMPONENT OF THE VELOCITY VECTOR POTENTIAL FOR WHICH THE COEFFICIENT IS BEING COMPUTED.

(4,3,2,1,0) - ORDER OF THE DERIVATIVE OF THE ABOVE COMPONENT.

M(1,2) - 1 REFERS TO TERMS NOT CONTAINING GAMMA AS A FACTOR.

2 REFERS TO TERMS CONTAINING GAMMA AS A FACTOR.

SECTION II

RETURNS THE VALUES FOR THE COEFFICIENTS IN THE ARRAYS REPRESENTING THE CENTRAL DIFFERENCE APPROXIMATION OF THE VORTICITY TRANSPORT EQUATION USING THE COEFFICIENTS COMPUTED IN SECTION I.

DESCRIPTION OF PARAMETERS

K - INDICATES THE POSITION OF THE POINT IN THE FINITE DIFFERENCE MESH RELATIVE TO THE CENTRAL POINT IN THE CENTRAL DIFFERENCE SCHEME. IF THE CENTRAL DIFFERENCE IS BEING GENERATED ABOUT THE N-TH POINT IN THE FINITE DIFFERENCE MESH, THEN K=1 REFERS TO THE POINT N+2, AS MEASURED TOWARD THE WALL, K=2 REFERS TO THE POINT N+1, K=3 REFERS TO THE POINT N, K=4 REFERS TO THE POINT N-1, AND K=5 REFERS TO THE POINT N-2.

EXAMPLE OF THE CALLING ARGUMENT:

C(F,G,H) M(1,2) EI

C(F,G,H) - COMPONENT OF THE VELOCITY VECTOR POTENTIAL FOR WHICH THE COEFFICIENT IS BEING COMPUTED.

M(1,2) - 1 REFERS TO TERMS NOT CONTAINING GAMMA AS A FACTOR.
2 REFERS TO TERMS CONTAINING GAMMA AS A FACTOR.

EI - REFERS TO THE LINEAR COMBINATION OF THE FIRST AND THIRD EQUATIONS RESULTING FROM EXPRESSION OF THE VORTICITY TRANSPORT EQUATION IN TERMS OF THE VELOCITY VECTOR POTENTIAL.

USAGE

CHM1E1 MUST BE DECLARED COMPLEX*16 IN THE CALLING PROGRAM.

OTHER ROUTINES REQUIRED

NCNE

FUNCTION CHM1E1(K,R)
IMPLICIT COMPLEX*16 (A-H,O-Z)
COMMON / COEFNT / A,B,G,REY,DEL
REAL*8 REY,R,DEL

DEFINE THE RECURRING PARAMETERS T1(R) AND T2(R).

T1(R) = A**2+(B/R)**2
T2(R) = A*2D0*(1D0-R**2)-T1(R)/REY

SECTION I

COEFFICIENTS OF THE COMPONENT F.

CF4M1(R) = B/(R*REY)
CF3M1(R) = 2D0*B/(R**2*REY)
CF2M1(R) = (B/R)*((T1(R)-1D0/R**2)/REY-T2(R))
CF1M1(R) = (B/R**2)*(((1D0-3D0*B**2)/R**2+A**2)/REY-T2(R))


```

CFOM1(R) = (B/R)*({4D0*B**2/(REY*R**4)-T1(R)*T2(R)})
CF2M2(R) = B/R
CF1M2(R) = B/R**2
CFOM2(R) = B*T1(R)/R

```

COEFFICIENTS OF THE COMPONENT G.

```

CG2M1(R) = -4D0*A*B/(REY*R**2)
CG1M1(R) = 4D0*A*B/(REY*R**3)
CGOM1(R) = (2D0*A*B/R**2)*({T2(R)-(T1(R)+2D0/R**2)/REY})
CGOM2(R) = -2D0*A*B/R**2

```

COEFFICIENTS OF THE COMPONENT H.

```

CH4M1(R) = -A/REY
CH3M1(R) = -2D0*A/(R*REY)
CH2M1(R) = A*(T2(R)+(3D0/R**2-T1(R))/REY)
CH1M1(R) = (A/R)*({(3D0*(B**2-ID0)/R**2-A**2)/REY+T2(R)})
CHOM1(R) = A*(T1(R)*T2(R)+(T1(R)+3D0/R**2)/REY-T2(R))/R**2)
CH2M2(R) = -A
CH1M2(R) = -A/R
CHOM2(R) = A*(ID0/R**2-T1(R))

```

SECTION II

CENTRAL DIFFERENCE APPROXIMATION FOR COMPONENT H.

```

GC TO (11,12,13,14,15),K
11 CHM1E1 = CH4M1(R)/DEL**4+CH3M1(R)/(2D0*DEL**3)
GC TO 100
12 CHM1E1 = -4D0*CH4M1(R)/DEL**4-2D0*CH3M1(R)/(2D0*DEL**3)
* )+CH2M1(R)/DEL**2+CH1M1(R)/(2D0*DEL)
GC TO 100
13 CHM1E1 = 6D0*CH4M1(R)/DEL**4-2D0*CH2M1(R)/DEL**2
* +CHOM1(R)
GC TO 100
14 CHM1E1 = -4D0*CH4M1(R)/DEL**4+2D0*CH3M1(R)/(2D0*DEL**3)
* )+CH2M1(R)/DEL**2-CH1M1(R)/(2D0*DEL)

```



```

GC TO 100
15 CHM1E1 = CH4M1(R)/DEL**4-CH3M1(R)/(2D0*DEL**3)
100 RETURN

```

C

```

ENTRY CHM2E1(K,R)
GC TO {21,22,23,24,21},K
21 CHM2E1 = {0D0,0D0}
GC TO 200
22 CHM2E1 = CH2M2(R)/DEL**2+CH1M2(R)/(2D0*DEL)
GC TO 200
23 CHM2E1 = -2D0*CH2M2(R)/DEL**2+CH0M2(R)
GC TO 200
24 CHM2E1 = CH2M2(R)/DEL**2-CH1M2(R)/(2D0*DEL)
200 RETURN

```

CCCCC

CENTRAL DIFFERENCE APPROXIMATION FOR COMPONENT G.

```

ENTRY CGM1E1(K,R)
GC TO {31,32,33,34,31},K
31 CGM1E1 = {0D0,0D0}
GC TO 300
32 CGM1E1 = CG2M1(R)/DEL**2+CG1M1(R)/(2D0*DEL)
GC TO 300
33 CGM1E1 = -2D0*CG2M1(R)/DEL**2+CG0M1(R)
GC TO 300
34 CGM1E1 = CG2M1(R)/DEL**2-CG1M1(R)/(2D0*DEL)
300 RETURN

```

C

```

ENTRY CGM2E1(K,R)
GC TO {41,41,42,41,41},K
41 CGM2E1 = {0D0,0D0}
GC TO 400
42 CGM2E1 = CG0M2(R)
400 RETURN

```

CCCCC

CENTRAL DIFFERENCE APPROXIMATION FOR COMPONENT F.

```

ENTRY CFM1E1(K,R)
GC TO {51,52,53,54,55},K
51 CFM1E1 = CF4M1(R)/DEL**4+CF3M1(R)/(2D0*DEL**3)
GC TO 500
52 CFM1E1 = -4D0*CF4M1(R)/DEL**4-2D0*CF3M1(R)/(2D0*DEL**3)
* )+CF2M1(R)/DEL**2+CF1M1(R)/(2D0*DEL)

```



```

53 GC TO 500
   * CFM1E1 = 600*CF4M1(R)/DEL**4-2D0*CF2M1(R)/DEL**2
   *   +CF0M1(R)
54 GC TO 500
   * CFM1E1 = -4D0*CF4M1(R)/DEL**4+2D0*CF3M1(R)/(2D0*DEL**3
   *   +CF2M1(R)/DEL**2-CF1M1(R)/(2D0*DEL)
55 GC TO 500
   * CFM1E1 = CF4M1(R)/DEL**4-CF3M1(R)/(2D0*DEL**3)
500 RETURN
C
ENTRY CFM2E1(K,R)
GC TO (61,62,63,64,61),K
61 CFM2E1 = (0D0,0D0)
GC TO 600
62 CFM2E1 = CF2M2(R)/DEL**2+CF1M2(R)/(2D0*DEL)
GC TO 600
63 CFM2E1 = -2D0*CF2M2(R)/DEL**2+CF0M2(R)
GC TO 600
64 CFM2E1 = CF2M2(R)/DEL**2-CF1M2(R)/(2D0*DEL)
600 RETURN
END

```


.....FUNCTION CHM1E2(K,R).....
 (POLAR COORDINATES)

PURPOSE

SECTION I

RETURNS THE VALUES FOR EACH OF THE COEFFICIENTS IN THE SECOND EQUATION RESULTING FROM EXPRESSION OF THE VORTICITY TRANSPORT EQUATION IN TERMS OF THE VELOCITY VECTOR POTENTIAL.

DESCRIPTION OF PARAMETERS

EXAMPLE OF THE CALLING ARGUMENT:

C(F,G,H) (4,3,2,1,0) M(1,2)

C(F,G,H) - COMPONENT OF THE VELOCITY VECTOR POTENTIAL FOR WHICH THE COEFFICIENT IS BEING COMPUTED.

(4,3,2,1,0) - ORDER OF THE DERIVATIVE OF THE ABOVE COMPONENT.

M(1,2) - 1 REFERS TO TERMS NOT CONTAINING GAMMA AS A FACTOR.

2 REFERS TO TERMS CONTAINING GAMMA AS A FACTOR.

SECTION II

RETURNS THE VALUES FOR THE COEFFICIENTS IN THE ARRAYS REPRESENTING THE CENTRAL DIFFERENCE APPROXIMATION OF THE VORTICITY TRANSPORT EQUATION USING THE COEFFICIENTS COMPUTED IN SECTION I.

DESCRIPTION OF PARAMETERS

K - INDICATES THE POSITION OF THE POINT IN THE FINITE DIFFERENCE MESH RELATIVE TO THE CENTRAL POINT IN THE CENTRAL DIFFERENCE SCHEME. IF THE CENTRAL DIFFERENCE IS BEING GENERATED ABOUT THE N-TH POINT IN THE FINITE DIFFERENCE MESH, THEN K=1 REFERS TO THE POINT N+2, AS MEASURED TOWARD THE WALL, K=2 REFERS TO THE POINT N+1, K=3 REFERS TO THE POINT N, K=4 REFERS TO THE POINT N-1, AND K=5 REFERS TO THE POINT N-2.

EXAMPLE OF THE CALLING ARGUMENT:

```

C(F,G,H) M(1,2) E2
C(F,G,H) - COMPONENT OF THE VELOCITY VECTOR POTENTIAL FOR
           WHICH THE COEFFICIENT IS BEING COMPUTED.
M(1,2) - 1 REFERS TO TERMS NOT CONTAINING GAMMA AS A
          FACTOR.
          2 REFERS TO TERMS CONTAINING GAMMA AS A FACTOR.
E2 - REFERS TO THE SECOND EQUATION RESULTING FROM
     EXPRESSION OF THE VORTICITY TRANSPORT EQUATION IN
     TERMS OF THE VELOCITY VECTOR POTENTIAL.

```

USAGE

CHM1E2 MUST BE DECLARED COMPLEX*16 IN THE CALLING PROGRAM.

OTHER ROUTINES REQUIRED

NONE

```

FUNCTION CHM1E2(K,R)
IMPLICIT COMPLEX*16(A-H,O-Z)
COMMON / COEFNT / A,B,C,REY,DEL
REAL*8 REY,R,DEL

```

DEFINE THE RECURRING PARAMETERS T1(R) AND T2(R).

```

T1(R) = A**2+(B/R)**2
T2(R) = A*2D0*(1D0-R**2)-T1(R)/REY

```

SECTION I

COEFFICIENTS OF THE COMPONENT F.

```

CF3M1(R) = A/REY
CF2M1(R) = A/(R*REY)
CF1M1(R) = -A*(T2(R)+1D0/(REY*R**2))
CF0M1(R) = (2D0*B**2/R)*(2D0-A/(REY*R**2))

```



```

CF1M2(R) = A

      COEFFICIENTS OF THE COMPONENT G.

CG2M1(R) = -T1(R)/REY
CG1M1(R) = (B**2/R**2-A**2)/(R*REY)
CGOM1(R) = T1(R)*T2(K)+(A**2-B**2/R**2)/(REY*R**2)

CGCM2(R) = -T1(R)

      COEFFICIENTS OF THE COMPONENT H.

CH3M1(R) = B/(R*REY)
CH2M1(R) = 2D0*B/(R**2*REY)
CH1M1(R) = (-B/R)*(T2(R)+1D0/(REY*R**2))
CHOM1(R) = B*(((2D0*A**2+1D0/K**2)/REY-T2(R))/R**2-4D0*A)

CH1N2(R) = B/R
CHOM2(R) = B/R**2

```

SECTION II

CENTRAL DIFFERENCE APPROXIMATION FOR COMPONENT H.

```

GC TO (11,12,13,14,15),K
11 CHM1E2 = CH3M1(R)/(2D0*DEL**3)
GC TO 100
12 CHM1E2 = -2D0*CH3M1(R)/(2D0*DEL**3)+CH2M1(R)/DEL**2
  * +CH1M1(R)/(2D0*DEL)
GC TO 100
13 CHM1E2 = -2D0*CH2M1(R)/DEL**2+CHOM1(R)
GC TO 100
14 CHM1E2 = 2D0*CH3M1(R)/(2D0*DEL**3)+CH2M1(R)/DEL**2
  * -CH1M1(R)/(2D0*DEL)
GC TO 100
15 CHM1E2 = -CH3M1(R)/(2D0*DEL**3)
100 RETURN

      ENTRY CHM2E2(K,R)
GC TO (21,22,23,24,25),K
21 CHM2E2 = (0D0,0D0)

```



```

GC TO 200
22 CHM2E2 = CH1M2(R)/(2D0*DEL)
GC TO 200
23 CFM2E2 = CH0M2(R)
GC TO 200
24 CHM2E2 = -CH1M2(R)/(2D0*DEL)
200 RETURN

```

CCCC

CENTRAL DIFFERENCE APPROXIMATION FOR COMPONENT G.

```

ENTRY CGM1E2(K,R)
GC TO (31,32,33,34,31),K
31 CGM1E2 = (OD0,OD0)
GC TO 300
32 CGM1E2 = CG2M1(R)/DEL**2+CG1M1(R)/(2D0*DEL)
GC TO 300
33 CGM1E2 = -2D0*CG2M1(R)/DEL**2+CG0M1(R)
GC TO 300
34 CGM1E2 = CG2M1(R)/DEL**2-CG1M1(R)/(2D0*DEL)
300 RETURN

```

C

```

ENTRY CGM2E2(K,R)
GC TO (41,41,42,41,41),K
41 CGM2E2 = (OD0,OD0)
GC TO 400
42 CGM2E2 = CG0M2(R)
400 RETURN

```

CCCC

CENTRAL DIFFERENCE APPROXIMATION FOR COMPONENT F.

```

ENTRY CFM1E2(K,R)
GC TO (51,52,53,54,55),K
51 CFM1E2 = CF3M1(R)/(2D0*DEL**3)
GC TO 500
52 CFM1E2 = -2D0*CF3M1(R)/(2D0*DEL**3)+CF2M1(R)/DEL**2
* +CF1M1(R)/(2D0*DEL)
GC TO 500
53 CFM1E2 = -2D0*CF2M1(R)/DEL**2+CF0M1(R)
GC TO 500
54 CFM1E2 = 2D0*CF3M1(R)/(2D0*DEL**3)+CF2M1(R)/DEL**2
* -CF1M1(R)/(2D0*DEL)
GC TO 500
55 CFM1E2 = -CF3M1(R)/(2D0*DEL**3)
500 RETURN

```


C

```
        ENTRY CFM2E2(K,R)
        GO TO (61,62,61,64,61),K
        CFM2E2 = (OD0,OD0)
        GO TO 600
        61 CFM2E2 = CF1M2(R)/(2D0*DEL)
        GO TO 600
        62 CFM2E2 = -CF1M2(R)/(2D0*DEL)
        RETURN
        600 END
```



```

C.....
SUBROUTINE CDMTIN (CATEGORY F-1)
PURPOSE
    INVERT A COMPLEX*16 MATRIX
USAGE
    CALL CDMTIN(N,A,NDIM,DETERM)
DESCRIPTION OF PARAMETERS
    N - ORDER OF COMPLEX*16 MATRIX TO BE INVERTED
        (INTEGER) MAXIMUM 'N' IS 100
    A - COMPLEX*16 INPUT MATRIX (DESTROYED). THE
        INVERSE OF 'A' IS RETURNED IN ITS PLACE
    NDIM - THE SIZE TO WHICH 'A' IS DIMENSIONED
        (ROW DIMENSION OF 'A' ACTUALLY APPEARING
        IN THE DIMENSION STATEMENT OF USER'S
        CALLING PROGRAM)
    DETERM - COMPLEX*16 VALUE OF DETERMINANT OF 'A'
        RETURNED BY CDMTIN.
REMARKS
    MATRIX 'A' MUST BE A COMPLEX*8 GENERAL MATRIX
    IF MATRIX 'A' IS SINGULAR THAT MESSAGE IS PRINTED
    'N' MUST BE .LE. NDIM
SUBROUTINES AND FUNCTIONS REQUIRED
    ONLY BUILT-IN FORTRAN FUNCTIONS
METHOD
    GAUSSIAN ELIMINATION WITH COLUMN PIVOTING IS USED.
    THE DETERMINANT IS ALSO CALCULATED. A DETERMINANT
    OF ZERO INDICATES THAT MATRIX 'A' IS
    SINGULAR.
C.....
SUBROUTINE CDMTIN (N,A,NDIM,DETERM)
IMPLICIT REAL*8 (A-H,O-Z)

```

CTIN0480
CTIN0030
CTIN0040
CTIN0050
CTIN0060

CTIN0080
CTIN0090
CTIN0100

CTIN0120
CTIN0130
CTIN0140

CTIN0170

CTIN0200

CTIN0240

CTIN0270
CTIN0280
CTIN0290
CTIN0300

CTIN0320
CTIN0330
CTIN0340
CTIN0350
CTIN0360
CTIN0370
CTIN0380
CTIN0390

CTIN0500


```

C
C
C
      CCMPLX*16 A(NDIM,NDIM),PIVOT(100),AMAX,T,SWAP,
      DETERM,UPIVOT(100),INDEX(100,2)
      INTEGER*4 IPIVOT(100),INDEX(100,2)
      REAL*8 TEMP,ALPHA(100)

      INITIALIZATION
      DETERM = (1D0,0D0)
      DO 20 J=1,N
      ALPHA(J) = 0D0
      DO 10 I=1,N
      10 ALPHA(J)=ALPHA(J)+A(J,I)*DCCONJG(A(J,I))
      20 ALPHA(J)=DSQRT(ALPHA(J))
      20 IPIVOT(J)=0
      DO 600 I=1,N

      SEARCH FOR PIVOT ELEMENT
      AMAX = (0D0,0D0)
      DO 105 J=1,N
      IF (IPIVOT(J)-1) 60,105,60
      60 DO 100 K=1,N
      IF (IPIVOT(K)-1) 80,100,740
      80 TEMP=AMAX*DCCONJG(AMAX)-A(J,K)*DCCONJG(A(J,K))
      85 IF (TEMP) 85, 85, 100
      85 IRCW=J
      ICCLUM=K
      AMAX=A(J,K)
      100 CCNTINUE
      105 IPIVOT(ICOLUM)=IPIVOT(ICOLUM)+1

      INTERCHANGE ROWS TO PUT PIVOT ELEMENT ON DIAGONAL
      IF (IROW-ICOLUM) 140, 260, 140
      140 DETERM=DETERM
      DO 200 L=1,N
      SWAP=A(IROW,L)
      A(IROW,L)=A(ICOLUM,L)
      A(ICOLUM,L)=SWAP
      SWAP=ALPHA(IROW)
      ALPHA(IROW)=ALPHA(ICOLUM)
      ALPHA(ICOLUM)=SWAP
      200 INDEX(I,1)=IROW
      INDEX(I,2)=ICOLUM
      260 IPIVOT(I)=A(ICOLUM,ICOLUM)
      U=PIVOT(I)
      TEMP=PIVOT(I)*DCCONJG(PIVOT(I))

```

```

CTIN0530
CTIN0550
CTIN0560
CTIN0570
CTIN0590
CTIN0610
CTIN0640
CTIN0650
CTIN0660
CTIN0670
CTIN0680
CTIN0700
CTIN0710
CTIN0720
CTIN0730
CTIN0750
CTIN0760
CTIN0770
CTIN0780
CTIN0790
CTIN0800
CTIN0810
CTIN0820
CTIN0830
CTIN0840
CTIN0850
CTIN0860
CTIN0870
CTIN0880
CTIN0890
CTIN0900
CTIN0910
CTIN0920
CTIN0930
CTIN0940
CTIN0950
CTIN0960
CTIN0970

```



```

C      IF(TEMP) 330, 720, 330
C      .   DIVIDE PIVOT ROW BY PIVOT ELEMENT
C
330  A(ICOL,ICOL) = (10,000)
    DC 350 L=1,N
    U=PIVOT(I)
350  A(ICOL,L)=A(ICOL,L)/U
C      .
C      REDUCE NON-PIVOT ROWS
C
380  DC 550 L=1,N
    IF(L1-ICOL) 400, 550, 400
400  T=A(L1,ICOL)
    A(L1,ICOL) = (00,000)
    DC 450 L=1,N
    U=A(ICOL,L)
450  A(L1,L)=A(L1,L)-U*T
550  CCNTINUE
600  CCNTINUE
C      .
C      INTERCHANGE COLUMNS
C
620  DC 710 I=1,N
    L=N+1-I
    IF(INDEX(L,1)-INDEX(L,2)) 630, 710, 630
630  JROW=INDEX(L,1)
    JCOL=INDEX(L,2)
    DC 705 K=1,N
    SWAP=A(K,JROW)
    A(K,JROW)=A(K,JCOL)
    A(K,JCOL)=SWAP
705  CCNTINUE
710  CCNTINUE
    RETURN
720  WRITE(6,730)
730  FORMAT(20H  MATRIX IS SINGULAR)
740  RETURN
    END

```

```

CTIN0990
CTIN1000
CTIN1010
CTIN1020
CTIN1040
CTIN1050
CTIN1060
CTIN1070
CTIN1080
CTIN1090
CTIN1100
CTIN1110
CTIN1120
CTIN1140
CTIN1150
CTIN1160
CTIN1170
CTIN1180
CTIN1190
CTIN1200
CTIN1210
CTIN1220
CTIN1230
CTIN1240
CTIN1250
CTIN1260
CTIN1270
CTIN1280
CTIN1290
CTIN1300
CTIN1310
CTIN1320
CTIN1330
CTIN1340
CTIN1350
CTIN1360

```



```

C.....SUBROUTINE MULM(X1,X2,N,MDIM,TEMPV).....
C
C      PURPOSE
C
C      PERFORMS THE MATRIX MULTIPLICATION OF A BANDED MATRIX BY A
C      SQUARE MATRIX.  THE RESULT IS RETURNED IN MATRIX X1.
C
C      USAGE
C
C      CALL MULM(X1,X2,N,MDIM,TEMPV)
C
C      DESCRIPTION OF PARAMETERS
C
C      X1 - THE MULTIPLYING MATRIX ON INPUT AND THE RESULTANT PRODUCT
C           ON OUTPUT.
C
C      X2 - THE MULTIPLIED MATRIX.
C
C      N - THE ORDER OF X1 AND X2.
C
C      MDIM - THE DIMENSION OF X1 AND X2 FROM THE CALLING PROGRAM.
C
C      TEMPV - A WORKING VECTOR. MUST BE DIMENSIONED MDIM.
C
C      OTHER ROUTINES REQUIRED
C
C      NCNE
C.....
C      SUBROUTINE MULM(X1,X2,N,MDIM,TEMPV)
C      CCMPLX*16 X1(MDIM,MDIM), X2(MDIM,MDIM), TEMPV(MDIM), TEMP
C
C      STORE ROW I OF X1 IN TEMPV.
C
C      DO 100 I=1,N
C      DO 10 J=1,N
C      10 TEMPV(J) = X1(I,J)
C
C      MULTIPLY COLUMN J OF X2 BY ROW I OF X1 AND STORE IN X1(I,J).
C
C      DC 30 J=1,N
C      TEMP = (OD0,OD0)
C
C      CC 40 K=1,N
C
C

```



```
C      40 TEMP = TEMP + TEMPV(K)*X2(K,J)
      30 X1(I,J) = TEMP
      100 CONTINUE
      RETURN
      END
```


SUBROUTINE DSPLIT.....

PURPOSE

DSPLIT TAKES A MATRIX OF COMPLEX*16 NUMBERS AND SPLITS IT INTO TWO MATRICES, ONE CONTAINING THE REAL PART OF THE ORIGINAL MATRIX, AND ONE CONTAINING THE IMAGINARY PART.

USAGE

CALL DSPLIT(N,MDIM,A,AREAL,AIMAG)

DESCRIPTION OF PARAMETERS

- THE SIZE OF THE MATRIX A, AN N BY N SQUARE N MATRIX.

MDIM - THE COLUMN DIMENSION OF MATRIX A

A - THE INPUT MATRIX. MUST BE DIMENSIONED MDIM BY
AT LEAST N IN THE CALLING PROGRAM (COMPLEX#16)

AREAL, AIMAG - THE OUTPUT MATRICES CONTAINING THE
REAL AND IMAGINARY PARTS, RESPECTIVELY, OF
MATRIX A. MUST BE DIMENSIONED (MDIM, MDIM) IN THE
CALLING PROGRAM.

NOTES...

MATRIX A AND MATRIX AREAL MAY OVERLAP IF THEY ARE
DIMENSIONED IN THE CALLING PROGRAM AS FOLLOWS...

```
COMPLEX*16 A(MDIM,MDIM)
REAL*8 AREAL(MDIM,MDIM),AIMAG(MDIM,MDIM)
EQUIVALENCE(A(1,1),AREAL(1,1))
```

OTHER ROUTINES NEEDED

WZON

```

SUBROUTINE DSPLIT(N,MDIM,A,AR,AI)
REAL*8 A(2,MDIM,MDIM),AR(MDIM,MDIM),AI(MDIM,MDIM)

```



```

      DO 1 J=1,N
      DO 1 I=1,N
      AR(I,J) = A(1,I,J)
1  AI(I,J) = A(2,I,J)
      RETURN
      END

```

C


```

C.EHESSC.....D.....
FUNCTION
- REDUCTION OF A COMPLEX MATRIX TO COMPLEX
  UPPER HESSENBERG FORM.
USAGE
- CALL EHESSC(AR, AI, K, L, N, IA, ID)
PARAMETERS  AR      CN INPUT MATRIX OF DIMENSION N BY N.
                THE MATRIX TO BE REDUCED.
                ON OUTPUT CONTAINS THE REAL COMPONENTS
                OF THE REDUCED HESSENBERG FORM IN THE
                UPPER TRIANGULAR PORTION (INCLUDING MAIN
                AND SUB-DIAGONAL) AND THE DETAILS OF
                THE REDUCTION IN THE LOWER TRIANGULAR
                PORTION.
                AI     INPUT MATRIX OF DIMENSION N BY N
                CONTAINING THE IMAGINARY COUNTERPARTS
                TO AR, ABOVE.
                K      INPUT SCALAR CONTAINING THE ROW AND COLUMN
                INDEX OF THE STARTING ELEMENT TO BE
                REDUCED BY ROW SCALING. FOR UNBALANCED
                MATRICES SET K = 1.
                L      INPUT SCALAR CONTAINING THE ROW AND
                COLUMN INDEX OF THE LAST ELEMENT TO BE
                REDUCED BY ROW SCALING. FOR UNBALANCED
                MATRICES SET L = N.
                N      INPUT SCALAR CONTAINING THE ORDER OF
                THE MATRIX TO BE REDUCED.
                IA     INPUT SCALAR CONTAINING ROW DIMENSION
                OF AR AND AI IN THE CALLING PROGRAM.
                ID     OUTPUT VECTOR OF LENGTH L CONTAINING
                DETAILS OF THE TRANSFORMATIONS.
PRECISION
- SINGLE/DOUBLE
CODE RESPONSIBILITY
- T.J. AIRD/E.W. CHOU
LANGUAGE
- FORTRAN
.....
LATEST REVISION
- FEBRUARY 7, 1973
.....
SUBROUTINE EHESSC (AR, AI, K, L, N, IA, ID)
DIMENSION
DCUBBLE PRECISION
CCOMPLEX*16
EQUIVALENCE
1 DATA
LA=L-1
KPI=K+1
IF (LA .LT. KPI) GO TO 45

```

```

EHEC0010
EHEC0020
EHEC0030
EHEC0040
EHEC0050
EHEC0060
EHEC0070
EHEC0080
EHEC0090
EHEC0100
EHEC0110
EHEC0120
EHEC0130
EHEC0140
EHEC0150
EHEC0160
EHEC0170
EHEC0180
EHEC0190
EHEC0200
EHEC0210
EHEC0220
EHEC0230
EHEC0240
EHEC0250
EHEC0260
EHEC0270
EHEC0280
EHEC0290
EHEC0300
EHEC0310
EHEC0320
EHEC0330
EHEC0340
EHEC0350
EHEC0360
EHEC0370
EHEC0380
EHEC0390
EHEC0400
EHEC0410
EHEC0420
EHEC0440
EHEC0450
EHEC0460
EHEC0480
EHEC0490
EHEC0500

```



```

DO 40 M=KPL,LA
  I=M
  XR=ZERO
  XI=ZERO
  DO 5 J=M,L
    IF (DABS(AR(J,M-1))+DABS(AI(J,M-1)) .LE. DABS(XR)+DABS(XI))
      GO TO 5
    XR=AR(J,M-1)
    XI=AI(J,M-1)
    I=J
  CONTINUE
  ID(M)=I
  IF (I .EQ. M) GO TO 20

  INTERCHANGE ROWS AND COLUMNS OF
  ARRAYS AR AND AI

  MM1=M-1
  DO 10 J=MM1,N
    YR=AR(I,J)
    AR(I,J)=AR(M,J)
    AR(M,J)=YR
    YI=AI(I,J)
    AI(I,J)=AI(M,J)
    AI(M,J)=YI
  CONTINUE
  DO 15 J=1,L
    YR=AR(J,I)
    AR(J,I)=AR(J,M)
    AR(J,M)=YR
    YI=AI(J,I)
    AI(J,I)=AI(J,M)
    AI(J,M)=YI
  CONTINUE

  END INTERCHANGE
  IF (XR .EQ. ZERO .AND. XI .EQ. ZERO) GO TO 40
  MP1=M+1
  DO 35 I=MP1,L
    YR=AR(I,M-1)
    YI=AI(I,M-1)
    IF (YR .EQ. ZERO .AND. YI .EQ. ZERO) GO TO 35
    Y=Y/X
    AR(I,M-1)=YR
    AI(I,M-1)=YI
    DO 25 J=M,N
      AR(I,J)=AR(I,J)-YR*AR(M,J)+YI*AI(M,J)
      AI(I,J)=AI(I,J)-YI*AR(M,J)+Y*AR(M,J)
    CONTINUE
    DO 30 J=1,L
      AR(J,M)=AR(J,M)+YR*AR(I,J)-YI*AI(I,J)

```

EHEC0510
 EHEC0520
 EHEC0530
 EHEC0540
 EHEC0550
 EHEC0560
 EHEC0570
 EHEC0600
 EHEC0610
 EHEC0620
 EHEC0630
 EHEC0640
 EHEC0650
 EHEC0660
 EHEC0670
 EHEC0680
 EHEC0690
 EHEC0700
 EHEC0710
 EHEC0720
 EHEC0730
 EHEC0740
 EHEC0750
 EHEC0760
 EHEC0770
 EHEC0780
 EHEC0790
 EHEC0800
 EHEC0810
 EHEC0820
 EHEC0830
 EHEC0840
 EHEC0850
 EHEC0860
 EHEC0870
 EHEC0880
 EHEC0890
 EHEC0900
 EHEC0910
 EHEC0920
 EHEC0930
 EHEC0940
 EHEC0950
 EHEC0960
 EHEC0970
 EHEC0980
 EHEC0990
 EHEC1000

EHEC1010
EHEC1020
EHEC1030
EHEC1040
EHEC1050
EHEC1060

```
AI(J,M)=AI(J,M)+YR*AI(J,I)+YI*AR(J,I)
CONTINUE
CONTINUE
CONTINUE
CCCONTINUE
CCRETURN
CCEND
```

30
35
40
45


```

C
DIMENSION
DIMENSION
CCOMPLEX*16
DCUBLE PRECISION
DOUBLE PRECE
EQUIVALENCE
1
2
DATA
DATA
INFER=0
IER=0
DO 5 I=1,N
  IF (I .GE. K .AND. I .LE. L) GO TO 5
  WR(I)=HR(I,I)
  WI(I)=HI(I,I)
5 CONTINUE
NN=L
TR=ZERO
TI=ZERO
10 IF (NN .LT. K) GO TO 9005
  ITS=0
  NM1=NN-1
  IF (NN .EQ. K) GO TO 25
  SEARCH FOR NEXT EIGENVALUE
  LOOK FOR SINGLE SMALL SUB-DIAGONAL
  ELEMENTS
15 NPL=NN+K
  DO 20 LL=K,NM1
    M=NPL-LL
    NM1=M-1
    IF (DABS(HR(M,M)) + DABS(HI(M,M)) .LE. M) +
      FDELP*(DABS(HR(M,M)) + DABS(HI(M,M))) GO TO 30
1 2 CCATINUE
25 MEK
30 IF (M .EQ. NN) GO TO 110
  IF (ITS .EQ. 30) GO TO 115
  FORM SHIFT
  IF (ITS .EQ. 10 .OR. ITS .EQ. 20) GO TO 35
  SI=HR(NN,NN)
  XI=HI(NN,NN)
  XI=HR(NM1,NN)*HR(NN,NM1)-HI(NM1,NN)*HI(NN,NM1)
  IF (XR .EQ. ZERO .AND. XI .EQ. ZERO) GO TO 40
  YI=(HR(NM1,NM1)-SI)/TWO
  YI=(HI(NM1,NM1)-SI)/TWO
ELR10480
ELR10490
ELR10500
ELR10510
ELR10530
ELR10540
ELR10550
ELR10560
ELR10570
ELR10580
ELR10590
ELR10610
ELR10620
ELR10630
ELR10640
ELR10650
ELR10660
ELR10670
ELR10680
ELR10690
ELR10700
ELR10710
ELR10720
ELR10730
ELR10740
ELR10750
ELR10760
ELR10770
ELR10780
ELR10790
ELR10800
ELR10810
ELR10820
ELR10830
ELR10840
ELR10880
ELR10890
ELR10900
ELR10910
ELR10920
ELR10930
ELR10940
ELR10950
ELR10960
ELR10970
ELR10980
ELR10990
ELR11000

```



```

Z=CDSQRT(DCMLPX(YR**2-YI**2+XR,TWO*YR*YI+XI))
IF (YR*ZR+YI*ZI .LT. ZERO) Z=-Z
X=X/(Y+Z)
SR=SR-XR
SI=SI-XI
GO TO 40
35 SR=DABS(HR(NN,NM1))+DABS(HR(NM1,NN-2))
SI=DABS(HI(NN,NM1))+DABS(HI(NM1,NN-2))
40 CC 45 I=K,NN
HR(I,I)=HR(I,I)-SR
HI(I,I)=HI(I,I)-SI
45 CCATTINUE
TR=TR+SR
TI=TI+SI
ITS=ITS+1
C C
LOOK FOR TWO CONSECUTIVE SMALL
SUB-DIAGONAL ELEMENTS
XR=DABS(HR(NM1,NM1))+DABS(HI(NM1,NM1))
YR=DABS(HR(NN,NM1))+DABS(HI(NN,NM1))
ZR=DABS(HR(NN,NN))+DABS(HI(NN,NN))
NMJ=NM1-M
IF (NMJ .EQ. 0) GO TO 55
C C
FOR MM=NN-1 STEP -1 UNTIL M+1 DO
DC 50 J=1,NMJ
MM=NN-J
M1=MM-1
YI=YR
YR=DABS(HR(MM,M1))+DABS(HI(MM,M1))
XI=ZR
ZR=XR
XR=DABS(HR(M1,M1))+DABS(HI(M1,M1))
IF (YR.LE.RDELP*ZR/YI*(ZR+XR+XI)) GO TO 60
50 CCATTINUE
55 MM=M
C C
TRIANGULAR DECCOMPOSITION
60 MP1=MM+1
CC 85 I=MP1,NN
IM1=I-1
XR=HR(IM1,IM1)
XI=HI(IM1,IM1)
YR=HR(I,IM1)
YI=HI(I,IM1)
IF (DABS(XR)+DABS(XI) .GE. DABS(YR)+DABS(YI)) GO TO 70
DO 65 J=IM1,NN
ZR=HR(IM1,J)
HR(IM1,J)=HR(I,J)
HR(I,J)=ZR
C C

```


ELR11580
ELR11590
ELR11600
ELR11610
ELR11620
ELR11630
ELR11640
ELR11650
ELR11660
ELR11670
ELR11680
ELR11690
ELR11700
ELR11710
ELR11720
ELR11730
ELR11740
ELR11750
ELR11760
ELR11770
ELR11780
ELR11790
ELR11800
ELR11810
ELR11820
ELR11830
ELR11840
ELR11850
ELR11860
ELR11870
ELR11880
ELR11890
ELR11900
ELR11910
ELR11920
ELR11930
ELR11940
ELR11950
ELR11960
ELR11970
ELR11980
ELR11990
ELR12000
ELR12010
ELR12020
ELR12030
ELR12040
ELR12050

```

        ZI=HI(IM1,J)
        HI(IM1,J)=HI(I,J)
        HI(I,J)=ZI
        CONTINUE
65      Z=X/Y
        WR(I)=ONE
        GO TO 75
70      Z=Y/X
        WR(I)=-ONE
        HR(I,IM1)=ZR
75      HI(I,IM1)=ZI
        DO 80 J=I,NN
            HR(I,J)=HR(I,J)-ZR*HR(IM1,J)+ZI*HI(IM1,J)
            HI(I,J)=HI(I,J)-ZI*HR(IM1,J)
        CONTINUE
80      CONTINUE
85      COMPOSITION
        CC 105 J=MPL,NN
        JM1=J-1
        XHR=HR(J,JM1)
        XI=HI(J,JM1)
        HR(J,JM1)=ZERO
        HI(J,JM1)=ZERO
        IF (WR(J) .LE. ZERO) GO TO 95
90      DO 90 I=M,J
        ZR=HR(I,JM1)
        HR(I,JM1)=HR(I,J)
        HR(I,J)=ZR
        ZI=HI(I,JM1)
        HI(I,JM1)=HI(I,J)
        HI(I,J)=ZI
        CONTINUE
95      DO 100 I=M,J
        HR(I,JM1)=HR(I,JM1)+XHR*HR(I,J)-XI*HI(I,J)
        HI(I,JM1)=HI(I,JM1)+XI*HR(I,J)
        CONTINUE
100      CONTINUE
105      GO TO 15
        C 110 WR(NN)=HR(NN,NN)+TR
        WI(NN)=HI(NN,NN)+TI
        NN=NM1
        GO TO 10
        C 115 INFER=NN
        SET ERROR=NO CONVERGENCE TO AN
        EIGENVALUE AFTER 30 ITERATIONS

```


IER=129
9000 CCNTINUE
9005 CALL UERTST (IER,6HELRH1C)
RETURN
END

ELR12060
ELR12070
ELR12080
ELR12090
ELR12100


```

C      SUBROUTINE UERTST ( IER,NAME)
C-----LIBRARY 1-----
C      FUNCTION
C      USAGE
C      PARAMETERS      IER
C
C      NAME
C      N = INPUT VECTOR CONTAINING THE NAME OF THE
C      CALLING ROUTINE AS A SIX CHARACTER LITERAL
C      STRING.
C      LANGUAGE      - FORTRAN
C-----
C      LATEST REVISION      - JANUARY 18, 1974
C
C      SUBROUTINE UERTST( IER,NAME)
C      DIMENSION
C      INTEGER*2
C      EQUIVALENCE
C      DATA
C      * * * *
C      IER2=IER
C      IF ( IER2 .GE. WARN) GO TO 5
C
C      IER1=4
C      GO TO 20
C      IF ( IER2 .LT. TERM) GO TO 10
C
C      IER1=3
C      GO TO 20
C      IF ( IER2 .LT. WARF) GO TO 15
C
C      IER1=2
C      GO TO 20
C
C      IER1=1
C
C      IER2=IER2-IBIT(IER1)
C
C      - ERROR MESSAGE GENERATION
C      - CALL UERTST(IER,NAME)
C      - ERROR PARAMETER TYPE + N WHERE
C      TYPE= 128 IMPLIES TERMINAL ERROR
C      64 IMPLIES WARNING WITH FIX
C      32 IMPLIES WARNING
C      N = ERROR CODE RELEVANT TO CALLING ROUTINE
C      INPUT VECTOR CONTAINING THE NAME OF THE
C      CALLING ROUTINE AS A SIX CHARACTER LITERAL
C
C      ITYP(5,4),IBIT(4)
C      NAME(3)
C      WARN,WARF,TERM,PRINTR
C      (IBIT(1),WARN), (IBIT(2),WARF), (IBIT(3),TERM)
C      / ,WARN, ,ING, ,WITH, ,
C      ,WARN, ,ING, ,FIX, ,
C      ,TERM, ,INAL, ,
C      ,NON- , ,DEFI, ,NED , ,
C      / 32,64,128,0/
C      PRINTR / 6/
C      .GE. WARN) GO TO 5
C      NON-DEFINED
C
C      5
C      .LT. TERM) GO TO 10
C      TERMINAL
C
C      10
C      .LT. WARF) GO TO 15
C      WARNING(WITH FIX)
C
C      IER1=2
C      GO TO 20
C      WARNING
C
C      15
C      IER1=1
C      EXTRACT 'N'
C
C      20
C      IER2=IER2-IBIT(IER1)
C      PRINT ERROR MESSAGE
C
C-----

```



```

25  * WRITE (PRINTR,25) (ITYP(I,IER1),I=1,5),NAME,IER2,IER
    * FORMAT(' ** I M S L(UERTST) **',5A4,4X,3A2,4X,I2,
    * (IER = ',I3,',))
    * RETURN
    * END

```

```

UERT0490
UERT0500
UERT0510
UERT0520
UERT0530

```


LIST OF REFERENCES

1. Harrison, W. F., On the Stability of Poiseuille Flow, Aeronautical Engineer's Thesis, Naval Postgraduate School, 1975.

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